VIBRATIONS II

MULTIPLE DEGREE OF FREEDOM SYSTEM

General Approach

Simultaneous Differential Equations

Review/Summary

MULTIPLE DEGREE OF FREEDOM SYSTEM

General Solution Approach

The matrix equation of motion for a general multi-degree-of-freedom system can be written as (Time Domain):

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{f(t)\} \qquad (1)$$

The solution of this linear, constant matrix coefficient, second order differential equation follows the solution approach for the simpler single degree of freedom problem. The solution takes on the following form involving complementary and particular parts:

$$\{x(t)\} = \{x_c(t)\} + \left\{x_p(t)\right\} \qquad (2)$$

$\{x_c(t)\}$ ***is the complementary portion of the solution and depends on the system characteristics and initial conditions. The complementary portion of the solution is sometimes referred to as the transient portion of the solution.***

$\left\{x_p(t)\right\}$ ***is the particular portion of the solution and depends upon the system characteristics and harmonic forcing functions. The particular portion of the solution is sometimes referred to as the steady state portion of the solution.***

Frequently, one portion of the solution will be of interest due to the application under study. If both portions are of interest, the initial conditions must not be applied until the total solution is formed (both complementary and particular portions).

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary (Transient) Solution:

The complementary solution is found by transforming the original differential equation, in homogeneous form (temporarily removing the forcing function(s)), from the time domain to the frequency domain (transforming from differential to algebraic equations) by using Fourier transforms, Laplace transforms or by assuming a solution that is appropriate for the second order, linear, constant coefficient matrix differential equation.

The system of equations has a complementary solution of the assumed form:

$$\{x_c(t)\} = \{X\}\ e^{s\,t} \tag{3}$$

$$\{\dot{x}_c(t)\} = s\ \{X\}\ e^{s\,t} = s\ \{x_c(t)\} \tag{4}$$

$$\{\ddot{x}_c(t)\} = s^2\ \{X\}\ e^{s\,t} = s^2\ \{x_c(t)\} \tag{5}$$

where:

- $s = \sigma + j\ \omega$ ***= complex valued frequency***

Note that the derivative of the response vector is simply the same vector multiplied by the complex frequency.

Substituting the above assumed relationships into Equation 1:

$$[M\ ]\ s^2\ \{X\ \}\ e^{s\,t} + [C\ ]\ s\ \{X\ \}\ e^{s\,t} + [K\ ]\ \{X\ \}\ e^{s\,t} = \{0\ \} \tag{6}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach

The non-trivial solution of the above equation leaves:

$$\left[s^2[M\ ] + s[C\ ] + [K\ ] \right] \{X\ \} = \{0\ \} \tag{7}$$

The above equation can be solved in terms of $2N$ characteristic values and $2N$ characteristic vectors. Due to the underdamped nature of vibration problems, there will always be N pairs of characteristic values and N pairs of characteristic vectors. The characteristic values are the complex-valued natural frequencies (modal frequencies) λ_r and the characteristic vectors are the complex-valued modal vectors $\{\psi_r\}$. Since the characteristic values and vectors are often found by placing the previous equation in an eigenvalue form, the characteristic values are often referred to as eigenvalues and the characteristic vectors as eigenvectors.

More commonly, the characteristic values and vectors are found by way of rudimentary mathematic manipulations as follows:

The characteristic values can be found for the above algebraic equation, for non-trivial solutions of $\{X\}$, from the matrix characteristic equation as follows:

$$\left| s^2[M\ ] + s[C\ ] + [K\ ] \right| = 0 \tag{8}$$

The above matrix characteristic equation is of model order two (2) with coefficient matrices of size N ($N \times N$). Therefore, this characteristic equation will yield $2N$ modal frequencies.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach

Expanding the matrix characteristic equation completely yields a high order ($2N$) polynomial characteristic equation with scalar coefficients:

$$\alpha_{2N}\ s^{2N} + \alpha_{2N-1}\ s^{2N-1} + \alpha_{2N-2}\ s^{2N-2} + \ldots\ldots +$$

$$\ldots\ldots + \alpha_2\ s^2 + \alpha_1\ s^1 + \alpha_0\ s^0 = 0$$

The characteristic values (complex valued modal frequencies ($\lambda_r = \sigma_r + j\omega_r$)) are found as the roots of this characteristic, high order ($2N$), scalar polynomial.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach

Once the modal frequencies (λ_r) have been determined, the characteristic vectors (modal vectors $\{\psi_r\}$) can be found from the following relationship:

$$\left[s^2[M\ ] + s[C\ ] + [K\ ] \right] \{X\ \} = \{0\ \} \tag{9}$$

Evaluating at $s = \lambda_r$:

$$\left[\lambda_r{}^2[M\ ] + \lambda_r[C\ ] + [K\ ] \right] \{\psi_r\ \} = \{0\ \} \tag{10}$$

Note that this system of linear equations is always rank deficient by at least one since the equation system is being evaluated at one of the characteristic frequencies (λ_r).

This process must be repeated for each modal frequency to determine each modal vector.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach

Once the modal frequencies (complex valued, in general) and modal vectors (complex valued, in general) are determined, the final form of the complementary solution can be formulated as follows:

$$\{x_c(t)\} = \sum_{r=1}^{N} \alpha_r \; \{\psi_r\} \; e^{\lambda_r \, t} + \alpha_r^* \left\{\psi_r^*\right\} e^{\lambda_r^* \, t} \tag{11}$$

Note that, in the above equation, the unknown coefficients α_r appear in complex conjugate pairs. This will always be true in the under-damped case. It is not necessary to assume that the conjugate relationship exists; this will result when the solution method is followed.

If there is no forcing function (the particular solution is zero), the unknown (complex valued) coefficients in the above equation (α_r) can be determined by applying the initial conditions to the above equation and/or the derivative of the above equation. The derivative of the above equation is shown below.

$$\{\dot{x}_c(t)\} = \sum_{r=1}^{N} \alpha_r \; \lambda_r \; \{\psi_r\} \; e^{\lambda_r \, t} + \alpha_r^* \; \lambda_r^* \left\{\psi_r^*\right\} e^{\lambda_r^* \, t} \tag{12}$$

If there is a forcing function, the solution for the unknown (complex-valued) coefficients must wait until the particular solution has been found. The initial conditions apply to the complete solution and the unknown coefficients can be found after the particular solution has been added to the complementary solution.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Particular Solution Approach

The particular solution is found by assuming a solution form for the response consistent with the forcing function characteristic. Since the forcing function (steady-state) is some form of harmonic (sine plus cosine terms), the forcing function and associated response can always be put into the following form (use the Euler identity for sine and cosine). This approach to solving for the particular solution is known as the method of undetermined coefficients.

$$f(t) = A\cos(\omega_a\, t) + B\sin(\omega_a\, t) \tag{13}$$

$$f(t) = A\left(\frac{e^{j\omega_a t} + e^{-j\omega_a t}}{2}\right) + B\left(\frac{e^{j\omega_a t} - e^{-j\omega_a t}}{2j}\right) \tag{14}$$

$$f(t) = A\left(\frac{e^{j\omega_a t} + e^{-j\omega_a t}}{2}\right) - jB\left(\frac{e^{j\omega_a t} - e^{-j\omega_a t}}{2}\right) \tag{15}$$

Collecting like terms:

$$f(t) = \left(\frac{A}{2} - j\,\frac{B}{2}\right)e^{j\omega_a t} + \left(\frac{A}{2} + j\,\frac{B}{2}\right)e^{-j\omega_a t} \tag{16}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Particular Solution Approach

Therefore, representing the characterisitics of the force at ω_a with a complex-valued magnitude yields the general form for the forcing function:

$$\{f(t)\} = \{F\}\, e^{j\omega_a t} + \left\{F^*\right\} e^{-j\omega_a t} \tag{17}$$

The response(s) to the previous forcing function(s) will be of the same form:

$$\left\{x_p(t)\right\} = \{X\}\, e^{j\omega_a t} + \left\{X^*\right\} e^{-j\omega_a t} \tag{18}$$

$$\left\{\dot{x}_p(t)\right\} = j\omega_a\, \{X\}\, e^{j\omega_a t} - j\omega_a \left\{X^*\right\} e^{-j\omega_a t} \tag{19}$$

$$\left\{\ddot{x}_p(t)\right\} = -\,\omega_a^2\, \{X\}\, e^{j\omega_a t} - \omega_a^2 \left\{X^*\right\} e^{-j\omega_a t} \tag{20}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Particular Solution Approach

Substituting the above relationships into Equation 1 and collecting like terms gives the following form for the positive frequency terms ($e^{j\omega_a t}$). Note that the portion of the solution involving the negative frequency terms ($e^{-j\omega_a t}$) is the complex conjugate of the positive frequency and provides no new information.

$$-\omega_a^2 [M] \{X\} + j\omega_a [C] \{X\} + [K] \{X\} = \{F\} \tag{21}$$

Note that the above linear, algebraic matrix equation involves N independent equations, as long as the system is not undamped with the forcing frequency ω_a equal to one of the natural frequencies. Since the complex-valued forcing vector $\{F\}$ provides N known pieces of information, this system of equations can be solved for the complex-valued response vector $\{X\}$.

Note also that, if more than one forcing frequency is present, the above particular solution process must be repeated for each forcing frequency.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Particular Solution Approach

An alternative to the traditional method of undetermined coefficients is the frequency response function approach. In this approach, the forcing function(s) are described, as above, in the frequency domain. The frequency response function(s) between the forcing degrees-of-freedom (DOFs) and the response degrees-of-freedom (DOFs) are computed from:

$$\left[H(\omega_a)\right] = \left[-\omega_a^2\,[M] + j\omega_a\,[C] + [K]\right]^{-1} \tag{22}$$

The responses caused by the forcing functions can now be found (in the frequency domain) by:

$$\{X(\omega_a)\} = \left[H(\omega_a)\right]\{F(\omega_a)\} \tag{23}$$

The particular response(s) can now be formulated in the time domain by converting the frequency domain information back to the time domain using the Euler identities (sine and cosine).

Note also that, if more than one forcing frequency is present, the above particular solution process must be repeated for each forcing frequency.

MULTIPLE DEGREE OF FREEDOM SYSTEM

Total Solution

The total solution, therefore, can be formulated as:

$$\{x(t)\} = \{x_c(t)\} + \left\{x_p(t)\right\} \tag{24}$$

$$\{x(t)\} = \sum_{r=1}^{N} \left(\alpha_r \ \{\psi_r\} \ e^{\lambda_r \ t} + \alpha_r^* \left\{\psi_r^*\right\} e^{\lambda_r^* \ t} \right) + \left(\ \{X\} \ e^{j\omega_a \ t} + \left\{X^*\right\} e^{-j\omega_a \ t} \ \right) \tag{25}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Example Solution:

For the following three degree of freedom system, formulate the complete solution for the following initial conditions and forcing function. Matlab Script v2_090.m can be used to solve for the modal frequencies and modal vectors for this problem.

$$[M]=\begin{bmatrix}10 & 0 & 0\\ 0 & 14 & 0\\ 0 & 0 & 12\end{bmatrix} \qquad [C]=\begin{bmatrix}50 & -30 & 0\\ -30 & 55 & -25\\ 0 & -25 & 25\end{bmatrix} \qquad [K]=\begin{bmatrix}5000 & -3000 & 0\\ -3000 & 5500 & -2500\\ 0 & -2500 & 2500\end{bmatrix}$$

Initial Conditions: ***Forcing Function:***

$$\{x(0)\}=\begin{Bmatrix}1.0\\ 0\\ 0\end{Bmatrix} \qquad \{\dot{x}(0)\}=\begin{Bmatrix}20.0\\ 0\\ 0\end{Bmatrix} \qquad \{f(t)\}=\begin{Bmatrix}0\\ -75\ Sin(30t)\\ 0\end{Bmatrix}$$

Complementary Solution Results:

$$\lambda_1 = -0.1848 + 6.0760i$$
$$\lambda_2 = -1.6417 + 18.0458i$$
$$\lambda_3 = -3.6795 + 26.8767i$$

$$\{\psi_1\}=\begin{Bmatrix}1.0000\\ 1.5435\\ 1.8763\end{Bmatrix} \qquad \{\psi_2\}=\begin{Bmatrix}1.0000\\ 0.5722\\ -0.9933\end{Bmatrix} \qquad \{\psi_3\}=\begin{Bmatrix}1.0000\\ -0.7863\\ 0.3105\end{Bmatrix}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Example Solution - Particular Part:

$$\left[s^2\,[M] + s\,[C] + [K]\right]\left\{X_p(s)\right\} = \{F(s)\}$$

Formulating the positive frequency portion of the solution at 30 rad/sec (s = j30 = 30i):

$$\begin{bmatrix} -4000.0+1500.0i & -3000.0-900.0i & 0 \\ -3000.0-900.0i & -7100.0+1650.0i & -2500.0-750.0i \\ 0 & -2500.0-750.0i & -8300.0+750.0i \end{bmatrix}\left\{X_p(30i)\right\} = \begin{Bmatrix} 0 \\ 37.50i \\ 0 \end{Bmatrix}$$

$$\left\{X_p(30i)\right\} = \begin{Bmatrix} -0.0041+0.0011i \\ 0.0036-0.0045i \\ -0.0016+0.0009i \end{Bmatrix}$$

Therefore:

$$\left\{x_p(t)\right\} = \begin{Bmatrix} -0.0041+0.0011i \\ 0.0036-0.0045i \\ -0.0016+0.0009i \end{Bmatrix} e^{j30t} + \begin{Bmatrix} -0.0041-0.0011i \\ 0.0036+0.0045i \\ -0.0016-0.0009i \end{Bmatrix} e^{-j30t}$$

Or:

$$\left\{x_p(t)\right\} = \begin{Bmatrix} -0.0082\,Cos(30t) - 0.0022\,Sin(30t) \\ 0.0072\,Cos(30t) + 0.0090\,Sin(30t) \\ -0.0032\,Cos(30t) - 0.0018\,Sin(30t) \end{Bmatrix}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Example Solution - Complementary Part:

While the complementary solution of the matrix differential equation for linear vibration problems always involves complex conjugate pair solutions, it is easier, when finding the numerical solution using Matlab, to ignore this and determine the solution without this assumption. Once you find the solution, a check for complex conjugate pairs serves as a verification that no other mistakes have occurred.

Therefore:

$$\{x(t)\} = \{x_c(t)\} + \left\{x_p(t)\right\}$$

$$\{x(t)\} = \sum_{r=1}^{6} \alpha_r \{\psi_r\}\ e^{\lambda_r\ t} + \begin{Bmatrix} -0.0041 + 0.0011i \\ 0.0036 - 0.0045i \\ -0.0016 + 0.0009i \end{Bmatrix} e^{j30t} + \begin{Bmatrix} -0.0041 - 0.0011i \\ 0.0036 + 0.0045i \\ -0.0016 - 0.0009i \end{Bmatrix} e^{-j30t}$$

Rewriting in matrix notation:

$$\{x(t)\} = \begin{bmatrix} e^{\lambda_1 t}\ \{\psi\}_1 & e^{\lambda_2 t}\ \{\psi\}_2 & e^{\lambda_3 t}\ \{\psi\}_3 & \{\cdots\} & e^{\lambda_6 t}\ \{\psi\}_6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{bmatrix}$$

$$+ \begin{Bmatrix} -0.0041 + 0.0011i \\ 0.0036 - 0.0045i \\ -0.0016 + 0.0009i \end{Bmatrix} e^{j30t} + \begin{Bmatrix} -0.0041 - 0.0011i \\ 0.0036 + 0.0045i \\ -0.0016 - 0.0009i \end{Bmatrix} e^{-j30t}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Example Solution - Complementary Part:

Taking the time derivative to get velocity:

$$\{\dot{x}(t)\} = \begin{bmatrix} \lambda_1 \, e^{\lambda_1 t} \, \{\psi\}_1 & \lambda_2 \, e^{\lambda_2 t} \, \{\psi\}_2 & \lambda_3 \, e^{\lambda_3 t} \, \{\psi\}_3 & \{\cdots\} & \lambda_6 \, e^{\lambda_6 t} \, \{\psi\}_6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{bmatrix}$$

$$+ \, j30 \begin{Bmatrix} -0.0041 + 0.0011i \\ 0.0036 - 0.0045i \\ -0.0016 + 0.0009i \end{Bmatrix} e^{j30t} - \, j30 \begin{Bmatrix} -0.0041 - 0.0011i \\ 0.0036 + 0.0045i \\ -0.0016 - 0.0009i \end{Bmatrix} e^{-j30t}$$

Evaluating displacement and velocity at t=0:

$$\begin{Bmatrix} 1.0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \{\psi_1\} & \{\psi_2\} & \{\psi_3\} & \{\psi_4\} & \{\psi_5\} & \{\psi_6\} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{bmatrix} + \begin{Bmatrix} -0.0082 \\ 0.0071 \\ -0.0031 \end{Bmatrix}$$

$$\begin{Bmatrix} 20.0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \lambda_1 \, \{\psi_1\} & \lambda_2 \, \{\psi_2\} & \lambda_3 \, \{\psi_3\} & \lambda_4 \, \{\psi_4\} & \lambda_5 \, \{\psi_5\} & \lambda_6 \, \{\psi_6\} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{Bmatrix} + \begin{Bmatrix} -0.0639 \\ 0.2722 \\ -0.0542 \end{Bmatrix}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Example Solution - Complementary Part:

$$\begin{Bmatrix} 1.0082 \\ -0.0071 \\ 0.0031 \\ 20.0639 \\ -0.2722 \\ 0.0542 \end{Bmatrix} = [A] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{Bmatrix}$$

Where:

$$[A] = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ -0.7863 & 0.5722 & 1.5435 & 1.5435 & 0.5722 & -0.7863 \\ 0.3105 & -0.9933 & 1.8763 & 1.8763 & -0.9933 & 0.3105 \\ -3.6795-26.8767i & -1.6417-18.0458i & -0.1848-6.0760i & -0.1848+6.0760i & -1.6417+18.0458i & -3.6795+26.8767i \\ 2.8932+21.1335i & -0.9394-10.3255i & -0.2852-9.3783i & -0.2852+9.3783i & -0.9394+10.3255i & 2.8932-21.1335i \\ -1.1425-8.3456i & 1.6307+17.9246i & -0.3467-11.4003i & -0.3467+11.4003i & 1.6307-17.9246i & -1.1425+8.3456i \end{bmatrix}$$

Therefore:

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_6 \end{Bmatrix} = \begin{Bmatrix} 0.2567+0.2265i \\ 0.1890+0.2246i \\ 0.0584+0.1902i \\ 0.0584-0.1902i \\ 0.1890-0.2246i \\ 0.2567-0.2265i \end{Bmatrix}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Adjoint Matrix*

An alternative approach to determining the modal vectors in the complementary solution involves evaluating the adjoint matrix of the system impedance matrix at the modal frequencies(λ_r). This method has two advantages: 1) Formulating the adjoint matrix effectively solves the set of linear equations one time rather than one set of linear equations for each modal frequency 2) Evaluating the adjoint matrix handles the rank deficient problem in a uniform way (no assumption of 1.0 for a modal coefficient). The development is as follows:

$$\left[s^2 [\ M\ ] s + s [\ C\ ] + [\ K\ ] \right] \{\ X\ \} = \{\ 0\ \}$$

Define:

$$[\ B(s)\ ] = \left[s^2 [\ M\ ] + s [\ C\ ] + [\ K\ ] \right]$$

where:

- $[\ B(S)\ ]$ = ***System Impedance Matrix***

$$[\ B(s)\ ][\ B(s)\ ]^{-1} = [\ I\ ]$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - Adjoint Matrix

$$[\ B(s)\ ]^{-1} = \frac{[\ B(s)\ ]^{A}}{[\ B(s)\ ]}$$

where:

- $[\ B(s)\ ]^{A}$ ***is the adjoint of matrix*** $[\ B(s)\ ]$**.**

$$[\ B(s)\ ][\ B(s)\ ]^{A} = [\ B(s)\ ][\ I\ ]$$

Note: $[\ B(\lambda_r)\ ] = 0$**.**

Evaluating at $s = \lambda_r$ ***gives:***

$$\Big[\ B(\lambda_r)\ \Big]\Big[\ B(\lambda_r)\ \Big]^{A} = [\ 0\ ]$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Adjoint Matrix*

Using any column of $[\ B(\lambda_r)\ ]^A$***, the*** $i-th$ ***column for example*** $\{\ B(\lambda_r)\ \}_i^A$***. Therefore:***

$$\left[\ B\ (\ \lambda_r\ )\ \right]\{\ B\ (\ \lambda_r\ )\ \}_i^A = \{\ 0\ \}$$

$$\left[\lambda_r^{\ 2}[M\ ] + \lambda_r[C\ ] + [K\ ]\right]\{\psi_r\ \} = \{0\ \}$$

Note that the columns of the adjoint matrix $[\ B(\lambda_r)\ ]^A$ ***are all proportional to the*** r***-th modal vector.***

When the mass, damping and stiffness matrices are symmetric (when absolute coordinates are used), the system impedance matrix $[\ B(s)\ ]$ ***is symmetric. Therefore, in this case, the adjoint matrix of*** $[\ B(\lambda_r)\ ]$ ***is also symmetric. Thus, in this case, the rows of the adjoint matrix are also proportional to the modal vector.***

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Eigenvalue/Eigenvector Method*

The homogeneous form of the Laplace domain model can be used as a general representation of the matrix relationship that yields the system modal characterisitics:

$$s^2[M]\{X\} + s[C]\{X\} + [K]\{X\} = \{0\}$$

Generally, this problem is solved using eigenvalue-eigenvector solution methods once the problem is put in the standard eigenvalue form:

$$[[A] - \lambda[I]]\{X\} = \{0\}$$

$$[A]\{X\} = \lambda\{X\}$$

$$[A]\{X\} = \lambda[B]\{X\}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Eigenvalue/Eigenvector Method*

Undamped Case:

In order to manipulate the system equations into a standard eigenvalue/eigenvector equation form, one approach is to assume that the undamped case is a reasonable approximation of the damped case. For many lightly damped situations, this is a reasonable assumption.

$$[K]\{X\} = -s^2[M]\{X\}$$

where:

- $\lambda = -s^2$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Eigenvalue/Eigenvector Method*

General Damped Case

$$[\ M\ ]\{\ \ddot{x}\ \} + [\ C\ ]\{\ \dot{x}\ \} + [\ K\ ]\{\ x\ \} = \{\ f\ \}$$

This system of equations can be augmented by the identity shown as follows:

$$[\ M\ ]\{\ \dot{x}\ \} - [\ M\ ]\{\ \dot{x}\ \} = \{\ 0\ \}$$

The above two equations can be combined to yield a new system of $2N$ ***equations. Note that all the matrices in the resulting equation are symmetric and the equation is now in a classical eigenvalue solution form. The notation used in the following equation is consistent with the notation used in many mathmatics and/or controls textbooks.***

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Eigenvalue/Eigenvector Method*

General Damped Case:

$$[\,A\,]\{\,\dot{y}\,\} + [\,B\,]\{\,y\,\} = \{\,f'\,\}$$

where:

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \qquad [B] = \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}$$

$$\{\,\dot{y}\,\} = \begin{Bmatrix} \{\,\ddot{x}\,\} \\ \{\,\dot{x}\,\} \end{Bmatrix} \qquad \{\,y\,\} = \begin{Bmatrix} \{\,\dot{x}\,\} \\ \{\,x\,\} \end{Bmatrix}$$

$$\{\,f'\,\} = \begin{Bmatrix} \{\,0\,\} \\ \{\,f\,\} \end{Bmatrix}$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - Eigenvalue/Eigenvector Method

General Damped Case

Forming the homogeneous equation of this system equation:

$$[\ A\ ]\{\ \dot{y}\ \} + [\ B\ ]\{\ y\ \} = \{\ 0\ \}$$

The solution of the above equation yields the complex-valued natural frequencies (eigenvalues) and complex-valued modal vectors (eigenvectors) for the augmented $2N$ ***equation system. Note that in this mathematical form, the eigenvalues will be found directly (not the square of the eigenvalue) and the*** $2N$ ***eigenvectors will be*** $2N$ ***in length. The exact form of the eigenvectors can be seen from the associated modal matrix for the*** $2N$ ***equation system. Note that the notation*** $\{\phi\}$ ***is used for an eigenvector in the*** $2N$ ***equation system and that the notation*** $\{\psi\}$ ***is used for an eigenvector of the*** N ***equation system.***

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - *Eigenvalue/Eigenvector Method*

General Damped Case:

MATLAB Solution

For the general, homogeneous system equation, using either an assumed solution or transform methods, the following equation must be solved:

$$\lambda [\ A\ ]\{\ y\ \} + [\ B\ ]\{\ y\ \} = \{\ 0\ \}$$

MATLAB uses the same matrix terminology but refers to a different eigenvalue equation.

$$[\ A\ ]\{\ y\ \} = \lambda [\ B\ ]\{\ y\ \}$$

Therefore, using the MATLAB EIG function to solve for the eigenvalues and eigenvectors requires the following form:

$$\left[\ y, \lambda\ \right] = EIG(B, -A)$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Complementary Solution Approach - Eigenvalue/Eigenvector Method

General Damped Case:

The eigenvalues of this system of equations are the same as for the original mass, stiffness and damping matrix equation. The eigenvectors of this system of equations yield the modal vectors of the original mass, stiffness and damping matrix equation through the modal matrix. The modal matrix for this system is a matrix made up of the 2N eigenvectors. The modal matrix $[\ \phi\ ]$ ***for this damped system can now be assembled.***

$$\left[\phi\right] = \left[\begin{array}{cccccc} \{\phi\}_1 & \{\phi\}_2 & \{\cdots\} & \{\phi\}_r & \{\cdots\} & \{\phi\}_{2N} \end{array}\right]$$

$$\left[\phi\right] = \left[\begin{array}{cccccc} \lambda_1\{\psi\}_1 & \lambda_2\{\psi\}_2 & \cdots & \lambda_r\{\psi\}_r & \cdots & \lambda_{2N}\{\psi\}_{2N} \\ \{\psi\}_1 & \{\psi\}_2 & \cdots & \{\psi\}_r & \cdots & \{\psi\}_{2N} \end{array}\right]$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Weighted Orthogonality Concept: Proportionally Damped Case

A set of weighted orthogonality relationships are valid for the system matrices $[M]$ ***and*** $[K]$***.***

$$\{\psi\}_r^T [M] \{\psi\}_s = 0$$

$$\{\psi\}_r^T [K] \{\psi\}_s = 0$$

Modal Scaling - Proportionally Damped Case

$$\{\psi\}_r^T [M] \{\psi\}_r = M_r$$

$$\{\psi\}_r^T [K] \{\psi\}_r = K_r$$

$$\{\psi\}_r^T [C] \{\psi\}_r = C_r$$

MULTIPLE DEGREE OF FREEDOM SYSTEM

Weighted Orthogonality Concept: General Case

A set of weighted orthogonality relationships are valid for the system matrices $[A]$ ***and*** $[B]$***.***

$$\{\phi\}_r^T [A] \{\phi\}_s = 0$$

$$\{\phi\}_r^T [B] \{\phi\}_s = 0$$

Modal Scaling - General Case

$$\{\phi\}_r^T [A] \{\phi\}_r = M_{A_r}$$

$$\{\phi\}_r^T [B] \{\phi\}_r = M_{B_r}$$

The terms *modal A* ***and*** *modal B* ***are modal scaling factors for the general case of system with damping. Note that modal A and modal B are related by the complex modal frequency for each mode.***