Appendix B: Singular Value Decomposition

The singular value decomposition will decompose a matrix into the simplest possible form, that being diagonal. This decomposition will always be possible regardless of the rank, or dimension, of the matrix [1].

Consider the right and left eigenvectors of the $m \times n$ matrix $[\, A \,]$, which is of rank k.

$$[\, A \,]\, \{v\} = \{u\}\, \sigma \tag{B-1}$$

$$[\, A \,]^H\, \{u\} = \{v\}\, \sigma \tag{B-2}$$

where:

- $\{u\} =$ $m \times 1$ left singular vector
- $\{v\} =$ $n \times 1$ right singular vector
- $\sigma =$ scalar singular value

By substituting Eq. (B-1) into Eq. (B-2) for $\{u\}$, the right singular vectors can be determined from:

$$[\, A \,]^H\, [\, A \,]\, \{v\} = \{v\}\, \sigma^2 \tag{B-3}$$

$$|\, [\, A \,]^H\, [\, A \,] - \sigma^2\, [\, I \,]\, | = 0$$

where:

- $i = 1 \rightarrow k \quad \sigma_i^2 > 0$
- $i = k+1 \rightarrow n \quad \sigma_i^2 = 0$
- $[\, V \,] = [\, \{v\}_1 \;\; \{v\}_2 \;\; \{v\}_3 \;\; \cdots \;\; \{v\}_n \,]$ right singular unitary matrix.

By substituting Eq. (B-2) into Eq. (B-1) for $\{v\}$, the left singular vectors can be determined from:

$$[\, A \,]\, [\, A \,]^H\, \{u\} = \{u\}\, \sigma^2 \tag{B-4}$$

$$| [A] [A]^H - \sigma^2 [I] | = 0$$

where:

- $i = 1 \rightarrow k \quad \sigma_i^2 > 0$
- $i = k+1 \rightarrow m \quad \sigma_i^2 = 0$
- $[U] = [\{u\}_1 \ \{u\}_2 \ \{u\}_3 \ \cdots \ \{u\}_m]$ left singular unitary matrix.

If the eigenvector matrices $[U]$ and $[V]$ are unitary ($[U]^H = [U]^{-1}$ and $[V]^H = [V]^{-1}$), that is, both columns and rows form an orthonormal set, the linear transformation will preserve both angles and lengths. Interpreted geometrically, linear transformations defined by unitary matrices behave like simple rotations in space. The matrix $[A]$, can now be decomposed into diagonal form by appending the eigenvector matrices to Eq. (B-1) and premultiplying by $[U]^H$, forming the matrix product:

$$[S] = [U]^H \ [A] [V] = \begin{bmatrix} \{u\}_1^H \\ \vdots \\ \{u\}_k^H \\ \vdots \\ \{u\}_m^H \end{bmatrix} \begin{bmatrix} \{u\}_1 \sigma_1 & \cdots & \{u\}_k \sigma_k & \{0\}_{k+1} & \cdots & \{0\}_n \end{bmatrix} \quad . \qquad \text{(B-5)}$$

Using the unitary matrix property, $[U]^H \ [U] = [I]$, the right hand side can be further simplified as:

$$[S] = [U]^H \ [A] [V] = \left[\begin{array}{ccccc|c} \sigma_1 & 0 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & 0 & \cdots & 0 & [0] \\ 0 & 0 & \sigma_3 & & 0 & \\ \vdots & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & \sigma_k & \\ - & - & - & - & - & - \\ & [0] & & & & [0] \end{array} \right] = \begin{bmatrix} \lceil \Sigma \rfloor & [0] \\ [0] & [0] \end{bmatrix} \quad . \qquad \text{(B-6)}$$

Thus, the origonal matrix, $[A]$, is decomposed into the matrix $[S]$, with the singular values on the diagonal, by the unitary matrices $[U]$ and $[V]$.

Then the singular value decomposition of [A] is defined by:

$$[A] = [U]\,[S]\,[V]^H \tag{B-7}$$

Noting the partitioning of the matrix [S] of Eq. (B-6), the unitary transformation matrices [U] and [V] can be partitioned in the same way to yield:

$$[A] = [U]\,[S]\,[V]^H = \begin{bmatrix} [U]_1 & [U]_2 \end{bmatrix} \begin{bmatrix} \lceil \Sigma \rfloor & [0] \\ [0] & [0] \end{bmatrix} \begin{bmatrix} [V]_1^H \\ [V]_2^H \end{bmatrix} \tag{B-8}$$

Thus, the singular value decomposition of [A] can be further reduced to:

$$[A] = [U]_1 \lceil \Sigma \rfloor [V]_1^H \tag{B-9}$$

where:

- $[U]_1 =$ left singular submatrix of size $m \times k$
- $\lceil \Sigma \rfloor =$ diagonal singular value matrix of size $k \times k$
- $[V]_1^H =$ right singular submatrix of size $k \times n$.

The (Moore-Penrose) generalized inverse (pseudoinverse) [1-3], $[A]^+$, of [A] is:

$$[A]^+ = [V] \begin{bmatrix} \lceil \Sigma \rfloor^{-1} & [0] \\ [0] & [0] \end{bmatrix} [U]^H = [V]_1 \lceil \Sigma \rfloor^{-1} [U]_1^H \quad .$$

where:

- $[A]^+ = n \times m$ generalized inverse of [A]
- $[V]_1 =$ right singular submatrix of size $n \times k$.
- $\lceil \Sigma \rfloor^{-1} =$ inverse of diagonal singular value matrix of size $k \times k$
- $[U]_1^H =$ left singular submatrix of size $k \times m$.

B.1 References

[1] Ben Nobel, James W. Daniel, **Applied Linear Algebra**, Prentice-Hall, Inc., 1977, pp. 323-337.

[2] Gibert Strang, **Introduction to Applied Mathematics**, Wellesley-Cambridge Press, 1986.

[3] Gibert Strang, **Linear Algebra and Its Applications**, Academic Press, 1980.