Polynomial, Non-Polynomial, and Orthogonal Polynomial Generating Functions for Nonlinear System Identification

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Abstract

Traditional methods for identifying models of nonlinear systems use integer power series to construct nonlinear feedback forces, which act together with the external forces on an appropriately chosen nominal linear system model. Two primary disadvantages to using ordinary polynomial series in practice are that nonlinear characteristics in engineered structures and material constitutive laws are generally not governed by integer power series; and limitations on measurement dynamic range restrict the number of terms in the series and hence the fidelity of the nonlinear model. This paper addresses these disadvantages by discussing and implementing non-integer power series, normalized polynomial series, and orthogonal polynomial series for nonlinear structural dynamic system identification. The first of these generating series can describe general nonlinear stiffness and damping characteristics, whereas the second and third types of series help to avoid poor numerical conditioning associated with ordinary integer power series.

1. Introduction

1.1 Motivation

There is a disconnect between traditional system identification practices using integer power polynomials and the way in which physical systems actually behave and are observed. For instance, real elements and systems are not necessarily governed by quadratic, cubic, and higher-order polynomial laws; therefore, generating functions for estimating system model parameters should not always be restricted to the set of integer power polynomials. This is especially true in the case of material or element stiffness and damping laws, which are notoriously nonlinear and rarely described exactly by integer polynomials (e.g. rubber mounts) [1].

Furthermore, the input and output measurements used to extract model parameters/coefficients usually contain artifacts of the data acquisition process (e.g. limited frequency resolution and limited dynamic range) that are very sensitive to the polynomial order. For instance, it is numerically difficult to accommodate polynomial series which contain both small and large powers in parameter estimation when working with real data because limited dynamic measurement range results in poor numerical conditioning of the “inverse” problem (i.e. high condition numbers) [2]. These practical issues must be resolved before accurate models of real-world systems can be realized.

1.2 Previous work

Other researchers have also recognized and examined these issues. Specifically, researchers in [3] discuss the use of integer power polynomials for nonlinear frequency domain system identification using the “reverse path” approach and provide some guidance on selecting polynomials as generating functions in parameter estimation when the measurements are contaminated with uncorrelated noise. Likewise, previous work in frequency domain modal parameter estimation [4]
has explored and implemented orthogonal power polynomials as a means to widen the frequency bandwidth for estimating parameters in models of linear vibrating systems.

The present research expands on the work in [3,4] by defining major issues associated with system identification using generating functions in nonlinear structural dynamic systems, citing examples of practical cases in which integer power polynomials are not adequate for parameter estimation of real-world systems, and giving guidelines for implementing other types of generating functions including non-integer power series and orthogonal (i.e. numerically stable) power polynomials to obtain more accurate models of nonlinear systems.

2. Generating Functions

2.1 The “inverse” problem

“Inverse” problems use input and/or output measurements to estimate parameters in a model of some kind (refer to Figure 1). Accurate solutions to these types of problems can be found when good models and good data are available. Models are thought of as good if they contain mathematical terms and generating functions which accurately describe the internal interactions between system components. Three types of generating functions (GFs) are listed in Figure 1: integer power polynomials, non-integer power series, and orthogonal power polynomials. All of these families of GFs should be used together in applications to build accurate nonlinear structural dynamics models.

Data is thought of as good if it accurately describes all of the system dynamics of interest. Since data and GFs are combined in the inverse problem to estimate model parameters, it is important to recognize that GFs should only be selected after careful consideration of the type and quality of the available data. This idea is demonstrated in Section 2.4.

The typical inverse problem in nonlinear system identification can be stated qualitatively as follows:

Given an assumed model form, a parameterization of some kind, and an appropriate set of generating functions, use input and/or output measurements to estimate the model parameters.

For example, the frequency domain model and appropriate generating functions for identifying a single degree of freedom (SDOF) system with asymmetric nonlinear stiffness are given by:

\[ Y(\omega) = H_L(\omega)(F(\omega) - \mu_1 Y_{n1}(\omega) - \mu_2 Y_{n2}(\omega)) \]

\[ Y_{n1}(\omega) = F[y^2] \]

\[ Y_{n2}(\omega) = F[y^3] \]

where \( F[\cdot] \) is the Fourier transform operator, \( H_L(\omega) \) is the frequency response function (FRF) of the underlying (nominal) linear system, \( F(\omega) \) and \( Y(\omega) \) are the Fourier transform of the input and output, respectively, \( Y_{n1}(\omega) \) and \( Y_{n2}(\omega) \) are the frequency domain generating functions, and \( \mu_1 \) and \( \mu_2 \) are the static nonlinear parameters.

2.2 Integer power polynomials

Integer power polynomials and polynomial series can often be used to accurately describe nonlinearities in structural dynamic systems; however, there are classes of nonlinearity that cannot be modeled accurately even with infinitely many integer polynomial terms. For instance, consider the system in Figure 2 with combined...
viscous and Coulomb damping, which is representative of many types of hydraulic/pneumatic systems with moving mechanical parts (e.g. piston-cylinder). This system can be modeled with the following ordinary differential equation:

\[ m \ddot{y} + c \dot{y} + ky + f_n(\dot{y}) = f(t) \quad (4) \]

\[ f_n(\dot{y}) = \mu \text{sgn} (\dot{y}) \quad (5) \]

or in the steady state with the frequency domain equation:

\[ Y(\omega) = H_L(\omega)(F(\omega) - F_n(\omega)) \quad (6) \]

\[ H_L(\omega) = \frac{1}{k - m\omega^2 + j\omega c} \quad (7) \]

where \( m = 1 \) kg, \( c = 1 \) Ns/m, \( k = 1000 \) N/m, \( \mu = 10 \) N, and \( F_n(\omega) = F[f_n(\dot{y})] \).

Estimates of the underlying linear FRF and nonlinear parameters can be obtained using the Nonlinear Identification through Feedback of the Outputs (NIFO) technique [5]. The estimated linear FRF estimates are shown in Figure 3 along with the nonlinear FRFs and true underlying linear FRFs for comparison. The top plot clearly shows that the cubic polynomial GF does not adequately describe the nonlinear Coulomb friction characteristic. In fact, there is no integer polynomial series that does describe this characteristic because it is vertically tangent (“stiff”) near \( \dot{y} = 0 \); however, integer polynomials are horizontally tangent (“compliant”) near \( \dot{y} = 0 \).

\[ \frac{d(\text{sgn} (\dot{y}))}{d\dot{y}}|_{\dot{y}=0} = \infty \quad \frac{d(\dot{y}^{\alpha})}{d\dot{y}}|_{\dot{y}=0} = 0 \quad (8) \]

There is clearly a need for higher fidelity GFs with vertical tangency near \( \dot{y} = 0 \).

2.3 Rational hyperbolic and progressive power series

Although integer power polynomials cannot be used to describe the Coulomb friction nonlinearity, rational hyperbolic power series do possess the correct kind of tangency and provide good estimates of the underlying linear FRF as is evident from the middle and bottom plots in Figure 3. In general, hyperbolic GFs for nonlinear structural dynamic systems are of the following form:
\[ \Delta y^\frac{m}{n} \]  

where \( \Delta y \) is the relative motion across the nonlinear element and \( n \) and \( m \) are integers such that \( m > n \).

More specifically, the hyperbolic GFs, \( \text{sgn}(\dot{y})|\dot{y}|^{\frac{1}{2}} \) and \( \text{sgn}(\dot{y})|\dot{y}|^{\frac{1}{4}} \), closely resemble the true nonlinear Coulomb friction function, \( \text{sgn}(\dot{y}) \), near \( \dot{y} = 0 \) for the SDOF system in Eq. (4). Furthermore, note that \( \text{sgn}(\dot{y})|\dot{y}|^{\frac{1}{2}} \) provides a better estimate of \( H_L(\omega) \) because it more accurately captures the sharp discontinuity of the signum function near \( \dot{y} = 0 \).

### 2.3.1 Vehicle suspension as an example

The use of rational hyperbolic or progressive GFs is not only of academic interest but is also significant in real-world applications of nonlinear structural dynamic system identification. For example, coil springs and rubber end stops (bushings) are often installed in series in automobile suspensions to provide progressive (i.e., hardening) stiffness characteristics in order to counteract the sensible decreases in effective natural frequencies that occur when vehicles carry more cargo [6]. These progressive stiffness components can sometimes be modeled most accurately using rational progressive power series of the form,

\[ \Delta y^{p+\frac{m}{n}} \]  

where \( p, n, \) and \( m \) are integers such that \( m > n \).

To illustrate the use of non-integer power series in real-world applications, consider the flatbed trailer with leaf springs shown in Figure 4. Each tire patch was excited in the vertical direction over a frequency range from 0.5 to 35 Hz and the acceleration response was measured at each actuator wheel pan and each wheel spindle. The NIFO technique was then used to estimate the nominal linear FRFs for this system using three different generating functions to describe the modulation in frequency response: \( \Delta y^3 \), \( \text{sgn}(\Delta y)|\Delta y|^2 \), and \( \text{sgn}(\Delta y)|\Delta y|^{1.7} \). The results for one of the driving point FRFs at the right rear vehicle corner are shown in Figure 5. Note that the cubic and quadratic GFs do not adequately describe the nonlinear suspension characteristic; however, the progressive GF accurately describes this characteristic.

### 2.4 Orthogonal polynomials and conditioning

Although there are classes of structural dynamic nonlinearities that can be accurately described with one term polynomials GFs, many realistic
nonlinearities can only be described using multiple terms in polynomial series. Because these generating function series contain more than one polynomial term, the interactions between different terms in the series with regards to numerical limitations are important to consider. These interactions are similar to the ones encountered in frequency domain modal parameter estimation, where large frequency bands are numerically difficult to process.

There are two alternatives to ordinary polynomial GF series that help to alleviate numerical ill-conditioning: normalized GFs and pseudo-orthogonal GFs. Normalized GFs reduce the condition number (i.e. provide better conditioning) by constraining the range of the GF series, whereas pseudo-orthogonal GFs reduce the condition number by orthogonalizing the GF series.

For example, consider the following SDOF system with cubic and fifth-order stiffness:

\[
\begin{align*}
my'' + cy' + ky + f_n(y) &= f(t) \quad (11) \\
f_n(y) &= \mu_1 y^3 + \mu_2 y^5 \quad (12)
\end{align*}
\]

where \(m = 1 \text{ kg}, c = 1 \text{ Ns/m}, k = 1000 \text{ N/m}, \mu_1 = 1e4 \text{ N/m}^3, \) and \(\mu_2 = 5e5 \text{ N/m}^5\). Possible ordinary, normalized, and orthogonal GF sets for experimentally identifying the SDOF system in Eq. (11) are given below:

\[
\begin{align*}
GF_{\text{ordinary}} &= (\Delta y^3, \Delta y^5) \quad (13) \\
GF_{\text{normalized}} &= ((\Delta y_{\text{max}}/\Delta y)^3, (\Delta y_{\text{max}}/\Delta y)^5) \\
GF_{\text{orthogonal}} &= (4\Delta y^3, 16\Delta y^5 - 20\Delta y^3) \quad (15)
\end{align*}
\]

Note that a pseudo-orthogonal Chebyshev polynomial series has been used in Eq. (15). The first order term in \(\Delta y\) cannot be included in these series because it is an inherent part of the nominal linear FRF estimate.

The condition numbers of the inversion matrices using the NIFO technique for the three different kinds of GF series in Eqs. (13)-(15) are shown in the bottom of Figure 6. The top of this figure shows the linear, nonlinear, and estimated linear FRFs for the system. Although the orthogonal power series gives only marginal improvement in the condition number (factor of 5 reduction), the normalized series dramatically improves the conditioning of the inversion process (five orders of magnitude reduction).

This reduction in the condition number becomes very important in practical applications in which the proper form of the GFs are rarely, if ever, known \textit{a priori}. Moreover, orthogonal and especially normalized GF series provide more accurate estimates of the nominal linear FRFs and nonlinear parameters than ordinary series where finite numerical precision and data acquisition dynamic range are concerned.

3. Conclusions

The use of ordinary integer polynomials, non-integer power series, normalized polynomial series, and orthogonal polynomial series as generating functions in nonlinear structural dynamic system identification using inversion techniques has been discussed. In particular, practical and
numerical issues associated with these series have been addressed in the context of simulations and real-world applications. It was shown that non-integer power series must often be used to adequately describe nonlinear system characteristics and that normalized and orthogonal polynomial series provide substantial reductions in the condition number for the inversion.

References