Modal Parameter Estimation

Overview/Review

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Modal Parameter Estimation - Introduction

Topics

- ***Modal Parameter Estimation Concept***
- ***Nomenclature/Definitions***
- ***SDOF Methods***
  - ***Basic Equations***
  - ***Peak Pick***
  - ***Least Squares Frequency Domain Model***
- ***MDOF Methods***
  - ***Unified Matrix Polynomial Approach***
    - ***Educational View of All Algorithms***
    - ***Two Stage, Three Step Process***
  - ***Basic Equations***
  - ***First Stage - Modal Frequencies***
  - ***Second Stage - Modal Vectors and Scaling***
  - ***Model Order Determination***
- ***Algorithm Differences***
- ***Summary***

Modal Parameter Estimation - Concept

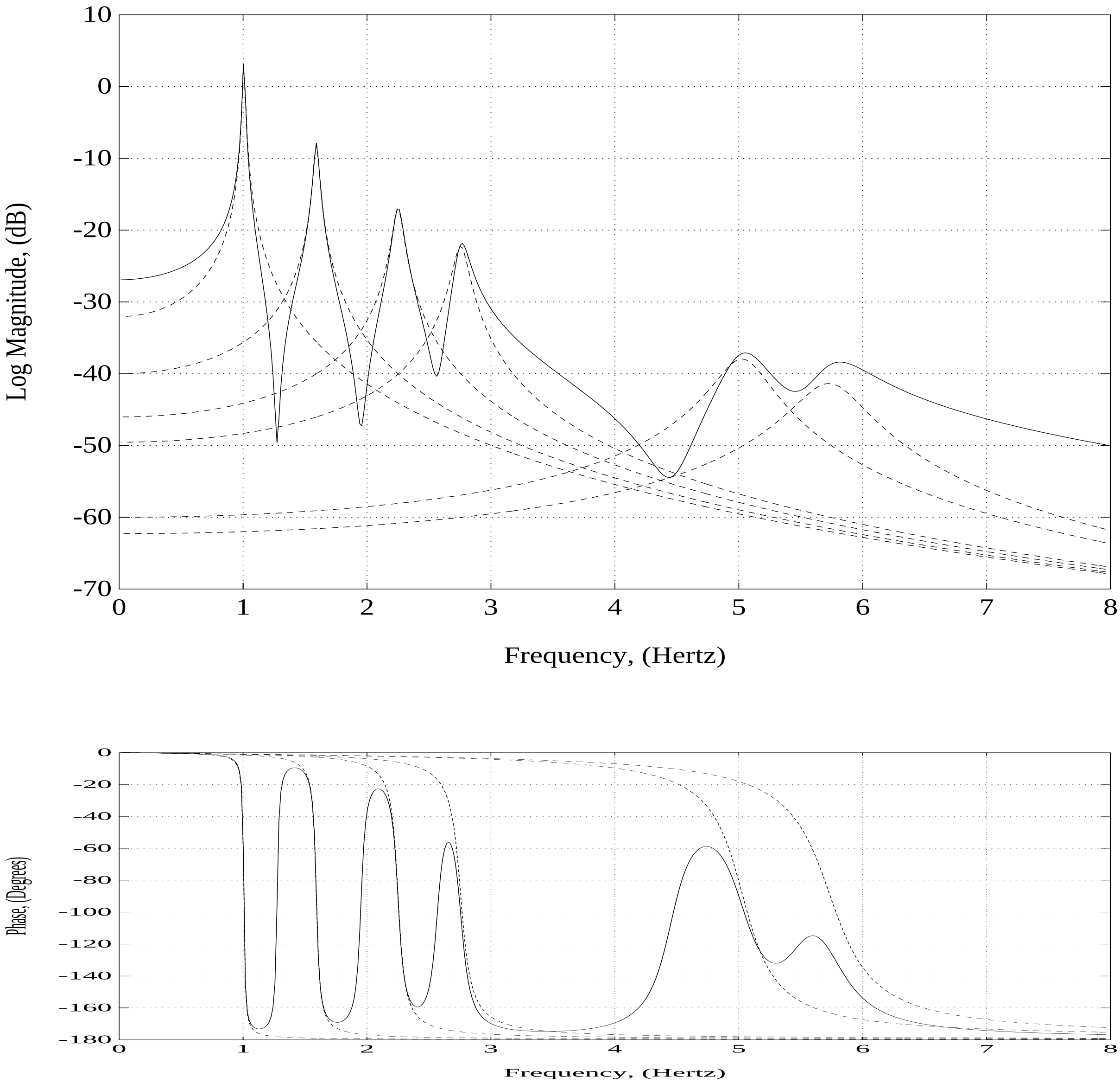


Figure 1. SDOF Contributions to Frequency Response Function

Modal Parameters - Definition

- ***Modal Frequencies (λ_r)***
    - ***Damped Natural Frequencies (ω_r)***
    - ***Damping Factors (σ_r)***
- ***Modal Vectors ($\{\psi_r\}$)***
    - ***Modal Participation Vectors ($\{L_r\}$)***
- ***Modal Scaling***
    - ***Modal Mass (M_r)***
    - ***Modal A (M_{A_r})***
    - ***Residues (A_{pqr})***

Development will be on the basis of displacement and force. Equations can be altered to velocity or acceleration via synthetic integration in the frequency response function formulation.

$$H_{pq}(\omega) = \frac{X(\omega)}{F(\omega)}$$

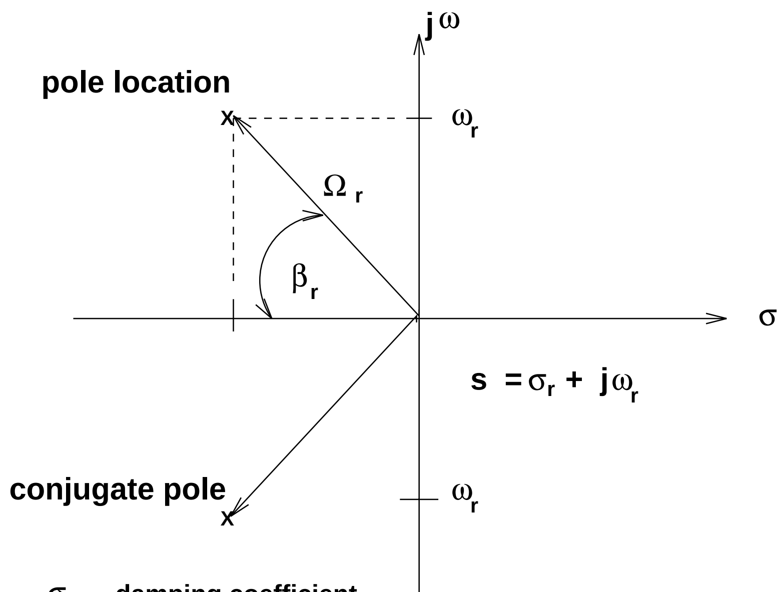
$$\frac{\dot{X}(\omega)}{F(\omega)} = (j\ \omega)\ \frac{X(\omega)}{F(\omega)}$$

$$\frac{\ddot{X}(\omega)}{F(\omega)} = (j\ \omega)^2\ \frac{X(\omega)}{F(\omega)}$$

Modal Parameter Estimation - Nomenclature

- ***s** = $\sigma + j\omega$ **= s Domain (Laplace) variable***
- *σ **= Angular damping variable (rad/sec)***
- *ω **= Angular frequency variable (rad/sec)***
- ***p = Measured degree of freedom (response)***
- ***q = Measured degree of freedom (input)***
- ***r = Modal vector number***
- ***2N = Number of modal frequencies (N complex conjugate pairs)***
- *N_o **= Number of output degrees of freedom (DOF)***
- *N_i **= Number of input degrees of freedom (DOF)***
- *λ_r **=** $\sigma_r + j\omega_r$ **= System pole = Modal frequency***
- *ψ_{pr} **= Modal coefficient for measured degree of freedom p and mode r***
- *ψ_{qr} **= Modal coefficient for measured degree of freedom q and mode r***
- *M_{A_r} **= Modal A scaling constant for mode r (Complex Modes)***
- *M_r **= Modal mass scaling constant for mode r (Normal Modes)***
- *A_{pqr} **= Residue for measured degree of freedom p, measured degree of freedom q and mode r***
- $A_{pqr} = \dfrac{\psi_{pr}\psi_{qr}}{M_{A_r}}$ ***(Complex Modes)***
- $A_{pqr} = \dfrac{\psi_{pr}\psi_{qr}}{j\,2\,M_r\,\omega_r}$ ***(Normal Modes)***

Modal Parameter Estimation - Nomenclature



σ_r - **damping coefficient**

ω_r **- damped natural frequency**

Ω_r **- resonant (undamped natural frequency)**

ζ_r **= cos** β_r **- damping factor (or percent of critical damping)**

Figure 2. Laplace Plane Pole Location

Modal Parameter Estimation - SDOF

Impulse Response Function Model - Principal Equation:

$$h_{pq}(t) = \sum_{r=1}^{N} A_{pqr}\ e^{\lambda_r t} + A^*_{pqr}\ e^{\lambda^*_r t}$$

SDOF Approximation (Mode r)

$$h_{pq}(t) \approx A_{pqr}\ e^{\lambda_r t} + A^*_{pqr}\ e^{\lambda^*_r t}$$

Frequency Response Function Model - Principal Equation:

Partial Fraction Model

$$H_{pq}(\omega) = \sum_{r=1}^{N} \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r}$$

Polynomial Model

$$H_{pq}(\omega) = \sum_{r=1}^{N} \frac{\psi_{pr}\ \psi_{qr}}{-M_r\ \omega^2 + j\ C_r\ \omega + K_r}$$

SDOF Approximation (Mode r)

$$H_{pq}(\omega) \approx \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r}$$

$$H_{pq}(\omega) \approx \frac{\psi_{pr}\ \psi_{qr}}{-M_r\ \omega^2 + j\ C_r\ \omega + K_r}$$

Modal Parameter Estimation - SDOF

Peak Pick Methods (Frequency Domain Method)

Modal Frequency ($\lambda_r = \sigma_r + j\ \omega_r$ ***)***

- ***Damped Natural Frequency (*** ω_r ***)***
  - ***Local Maxima/Minima in Frequency Response Functions***
  - ***Damped Natural Frequency, Peak Frequency, Undamped Natural Frequency Nearly Identical***
- ***Damping (*** σ_r ***)***
  - ***Log Decrement Estimate***
  - ***Half Power Estimate***

Modal Vector and Scaling

- ***Evaluate Frequency Response Function at Damped Natural Frequency***
- ***Fit to Theoretical Model (one data point)***

$$\{\ H(\omega_r)\ \} \approx \frac{\psi_{pr}}{j\ 2\ \omega_r\ M_r\ \sigma_r}\ \{\ \psi_r\ \}$$

$$\{\ H(\omega_r)\ \} \approx \frac{\psi_{pr}}{M_{A_r}}\ \{\ \psi_r\ \}$$

Modal Parameter Estimation - SDOF

Peak Pick - Concept

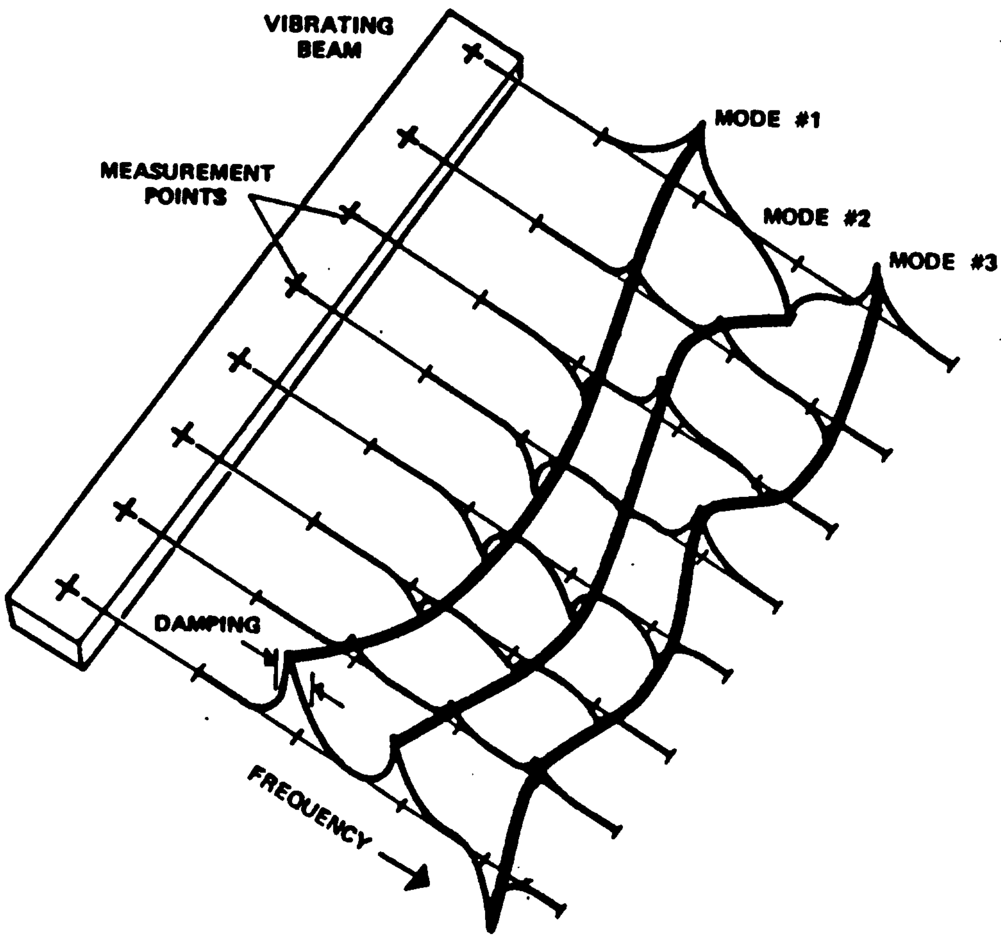
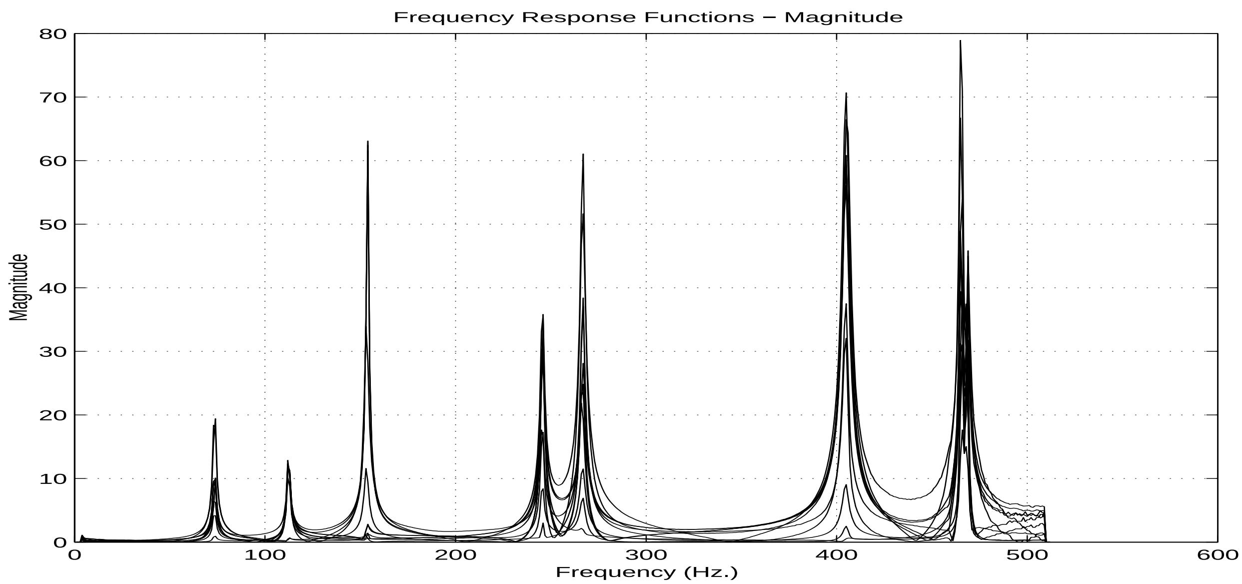


Figure 3. Imaginary Part of FRFs of Simple Free-Free Beam

Modal Parameter Estimation - SDOF

Peak Pick - Damped Natural Frequency



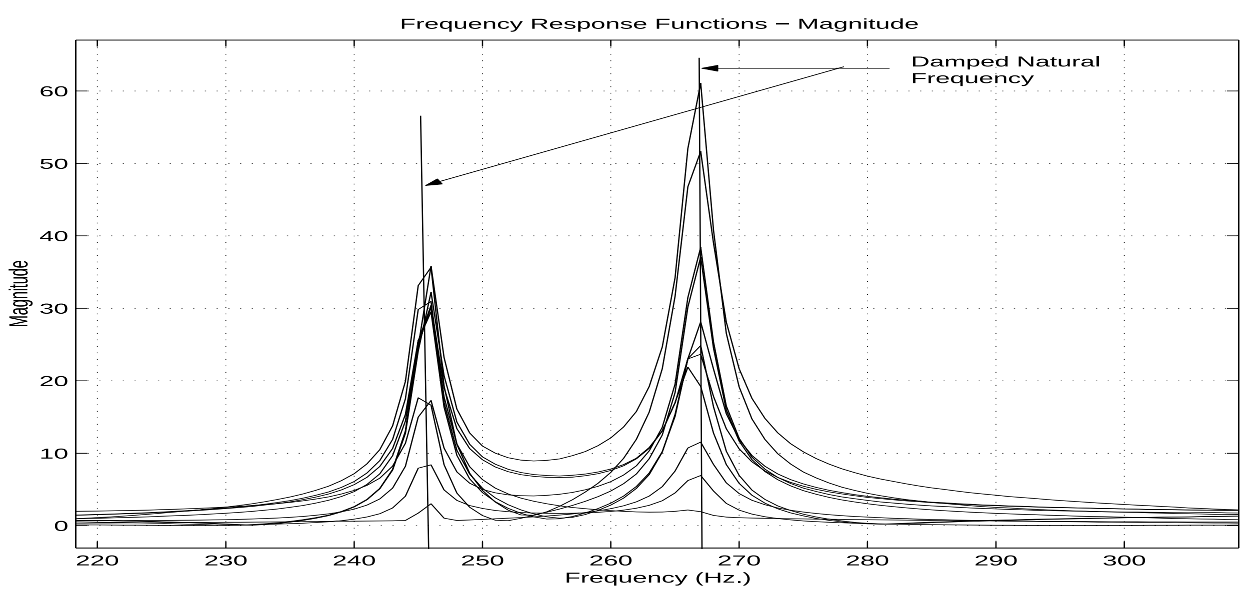


Figure 4. Damped Natural Frequencies at Maxima of FRF Magnitude

Modal Parameter Estimation - SDOF

Peak Pick - Damping (Half Power Method)

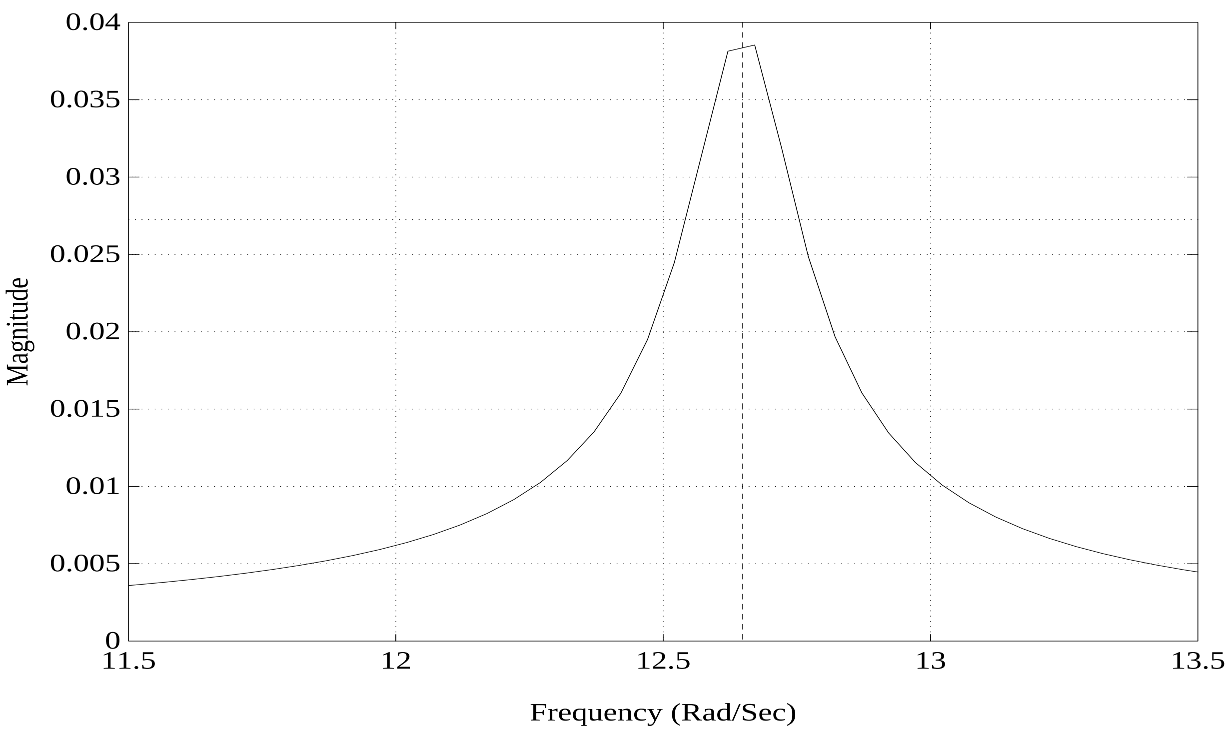


Figure 5. Damping Estimate from Half Power Method

Damping estimates from power methods require knowledge of full power. Even with accurate data, the discrete frequency sampling of the data may not yield full power information.

Modal Parameter Estimation - SDOF

Peak Pick - Modal Vectors

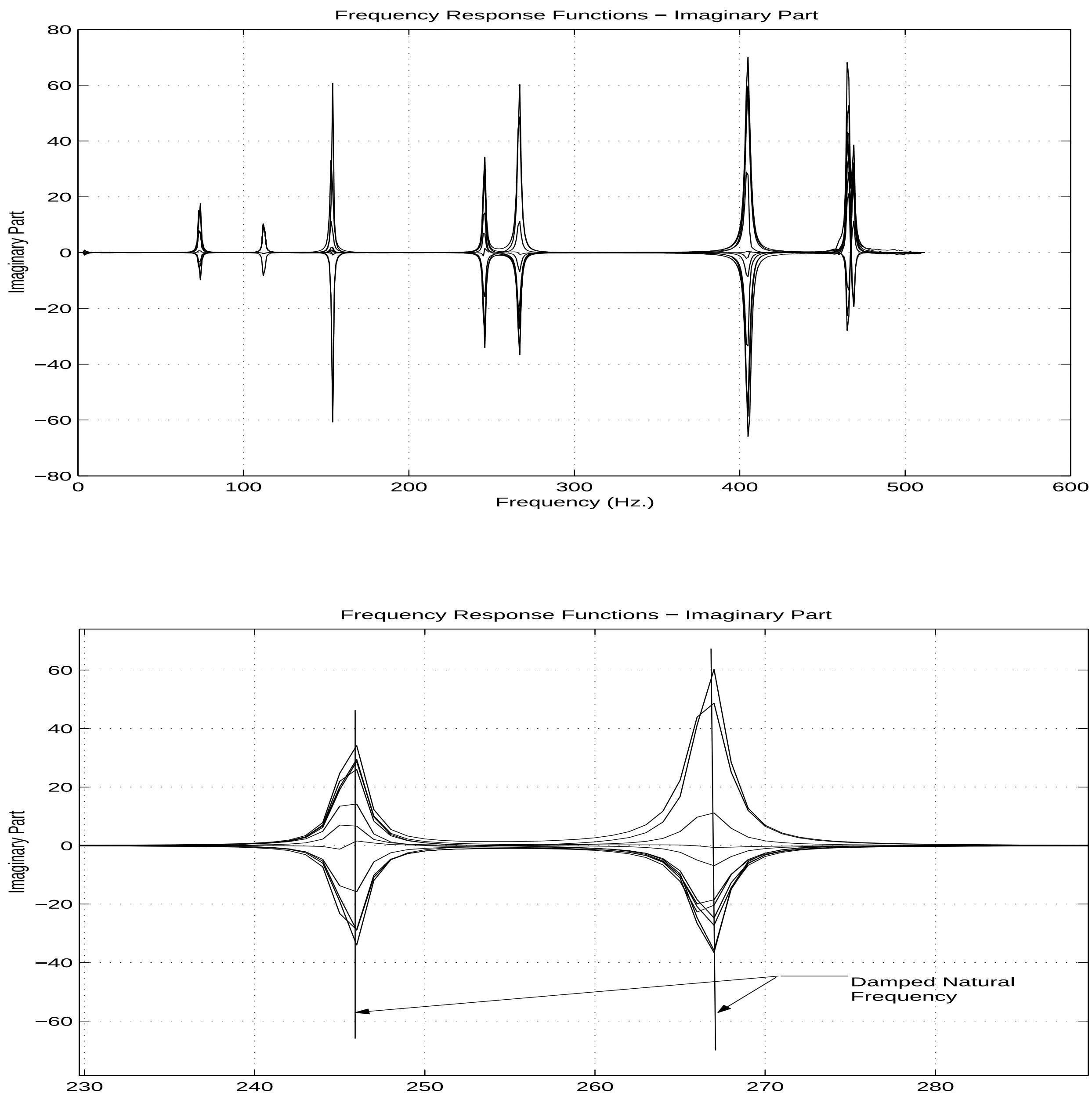


Figure 6. Damped Natural Frequencies - FRF Imaginary Part

Modal Parameter Estimation - SDOF

Least Squares Frequency Domain Models

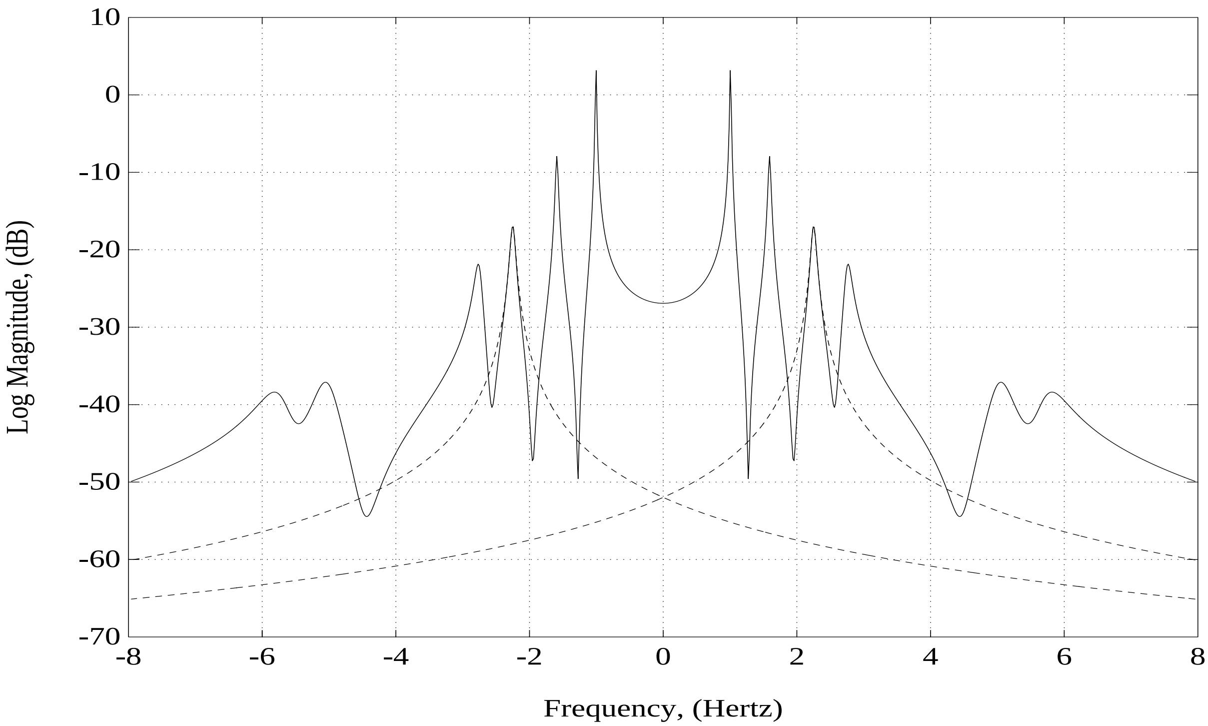
Many other SDOF methods have been developed based upon frequency domain models of the frequency response functions (FRFs) evaluated in the neighborhood of one of the damped natural frequencies. In every case, the modal parameters can be estimated based upon FRF information at several frequencies near the damped natural frequency that are normally identified by user interaction using the Peak Pick Method to define the frequency interval for each mode. When sufficient data is available, least squares methods are used to handle the overdetermined sets of equations.

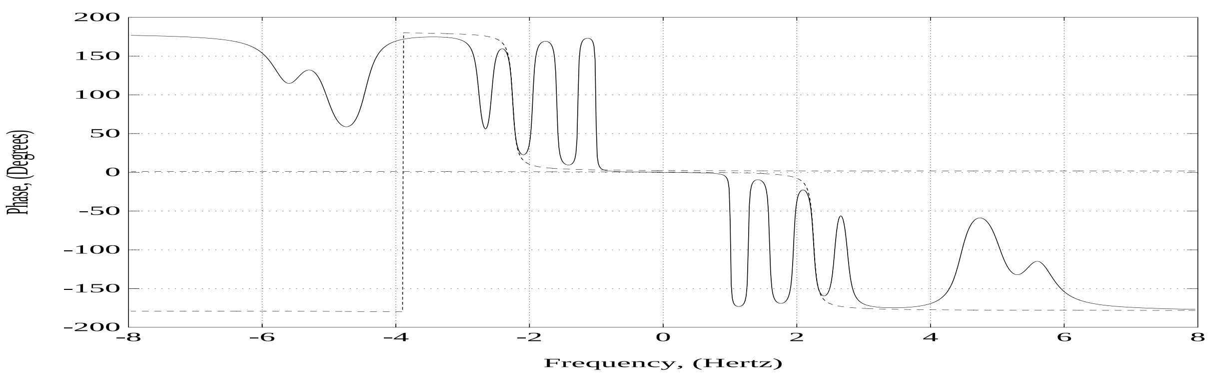
These methods include:

- ***Kennedy-Pancu Circle Fit***
- ***SDOF Polynomial Model***
- ***SDOF Partial Fraction Model***

Modal Parameter Estimation - SDOF

Partial Fraction Model Example





Modal Parameter Estimation - SDOF

Partial Fraction Model - Least Squares SDOF

Characteristics of Method

- ***Solves for both modal frequency (λ_r) and residue (A_{pqr}).***
- ***Uses the frequency response function information in the vicinity of a single mode.***
- ***Approximate method (ignores complex conjugate contribution).***
- ***Problem: Error when modal coefficient should be zero.***

The approximate relationship that is used in this case is represented in the following equation. The frequency ω_1 is a frequency near the damped natural frequency ω_r.

$$H_{pq}(\omega_1) \approx \frac{A_{pqr}}{j\omega_1 - \lambda_r}$$

$$H_{pq}(\omega_1)\ (j\omega_1 - \lambda_r) = A_{pqr}$$

$$H_{pq}(\omega_1)\ \lambda_r + A_{pqr} = (j\omega_1) H_{pq}(\omega_1)$$

Modal Parameter Estimation - SDOF

Partial Fraction Model - Least Squares SDOF

Repeating the previous equation for several frequencies in the vicinity of the peak frequency:

$$\begin{bmatrix} H_{pq}(\omega_1) & 1 \\ H_{pq}(\omega_2) & 1 \\ H_{pq}(\omega_3) & 1 \\ \cdots & 1 \\ H_{pq}(\omega_s) & \cdots \\ & 1 \end{bmatrix}_{N_s \times 2} \begin{Bmatrix} \lambda_r \\ A_{pqr} \end{Bmatrix}_{2 \times 1} = \begin{Bmatrix} (j\omega_1)\, H_{pq}(\omega_1) \\ (j\omega_2)\, H_{pq}(\omega_2) \\ (j\omega_3)\, H_{pq}(\omega_3) \\ \cdots \\ (j\omega_s)\, H_{pq}(\omega_s) \end{Bmatrix}_{N_s \times 1}$$

The above equation again represents an overdetermined set of linear equations that can be solved using any pseudo-inverse or normal equations approach.

Modal Parameter Estimation - SDOF

Conclusion/Summary of SDOF Methods

SDOF methods are fast, easy to implement and are always used as a first estimate for the modal vectors during the data acquisition phase of the testing. If the modes are lightly damped and well separated in frequency, SDOF methods may be all that is required in order to estimate modal parameters with acceptable accuracy.

Modal Parameter Estimation - MDOF

Current Identification Algorithm

	Domain		Matrix Polynomial Order			Coefficients	
Algorithm	Time	Freq	Zero	Low	High	Scalar	Matrix
CEA	•				•	•	
LSCE	•				•	•	
PTD	•				•		$N_i \times N_i$
ITD	•			•			$N_o \times N_o$
MRITD	•			•			$N_o \times N_o$
ERA	•			•			$N_o \times N_o$
PFD		•		•			$N_o \times N_o$
SFD		•		•			$N_o \times N_o$
MRFD		•		•			$N_o \times N_o$
RFP		•			•	•	Both
OP		•			•	•	Both
CMIF		•	•				$N_o \times N_i$

Modal Parameter Estimation Algorithms	
CEA	Complex Exponential Algorithm [1,2]
LSCE	Least Squares Complex Exponential [1,2]
PTD	Polyreference Time Domain [30,31]
ITD	Ibrahim Time Domain [9,12]
MRITD	Multiple Reference Ibrahim Time Domain [9]
ERA	Eigensystem Realization Algorithm [13,23]
PFD	Polyreference Frequency Domain [8,19-21,24,33]
SFD	Simultaneous Frequency Domain [3]
MRFD	Multi-Reference Frequency Domain [4]
RFP	Rational Fraction Polynomial [28]
OP	Orthogonal Polynomial [25,26,29]
CMIF	Complex Mode Indication Function [26]

Modal Parameter Estimation - MDOF

Unified Matrix Polynomial Algorithm (UMPA) Concept

The UMPA concept recognizes that both the time and frequency domain models generate essentially the same matrix polynomial model form. This does not mean that the alpha $[\alpha]$ and beta $[\beta]$ matrices are equal in the two cases (they are related) but the two corresponding models do yield exactly the same modal parameters.

It should be noted that, theoretically, these two models are not transform pairs but are derived from slightly different theoretical basis.

- ***Unified Mathematical Approach To Many Algorithms***
  - ***Educational Efficiency***
  - ***Numerical Comparisons***
  - ***Error Weighting/Sensitivity Evaluations***
- ***Emphasize Similarity of Algorithms***
- ***Emphasize General Concepts Not Specific Details***
- ***Algorithm Development Different From Original***

Modal Parameter Estimation - MDOF

Matrix Polynomial Characteristic Equation

Frequency Domain:

$$| [\alpha_m] \, s^m + [\alpha_{m-1}] \, s^{m-1} + [\alpha_{m-2}] \, s^{m-2} + \ldots\ldots\ldots + [\alpha_0] \, | = 0 \qquad (1)$$

Time Domain:

$$| [\alpha_m] \, z^m + [\alpha_{m-1}] \, z^{m-1} + [\alpha_{m-2}] \, z^{m-2} + \ldots\ldots\ldots + [\alpha_0] \, | = 0 \qquad (2)$$

$$z_r = e^{\lambda_r \, \Delta t} \qquad \lambda_r = \sigma_r + j \, \omega_r$$

$$\sigma_r = \mathrm{Re}\left[\frac{ln \; z_r}{\Delta t} \right] \qquad \omega_r = \mathrm{Im}\left[\frac{ln \; z_r}{\Delta t} \right]$$

Modal Parameter Estimation - MDOF

Unified Matrix Polynomial Algorithm (UMPA) Concept

All algorithms that can be represented by the UMPA concept involve two stages with three steps. The outline of this two stage, three step process is as follows:

First Stage

- ***First Step***
  - ***Load Measured Data into Linear Equation Form (Overdetermined).***
  - ***Find Scalar or Matrix Coefficients Using Least Squares Methodology (*** $[\alpha_k]$ ***and*** $[\beta_k]$***).***
- ***Second Step***
  - ***Use Scalar or Matrix Coefficients to Define Matrix Coefficient Polynomial (Characteristic) Equation.***
  - ***Solve Matrix Coefficient Polynomial Equation for Roots (Modal Frequencies).***

Second Stage

- ***Third Step***
  - ***Use Known Modal Frequencies with Measured Data in the Matrix Partial Fraction Equation to Form Linear Equations (Overdetermined).***
  - ***Find Residues and Modal Scaling Using Least Squares Methodology.***

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Single Input - Single Output Model

$$\frac{X_p(\omega)}{F_q(\omega)} = \frac{\beta_n\,(j\omega)^n + \beta_{n-1}\,(j\omega)^{n-1} + \cdots + \beta_1\,(j\omega)^1 + \beta_0\,(j\omega)^0}{\alpha_m\,(j\omega)^m + \alpha_{m-1}\,(j\omega)^{m-1} + \cdots + \alpha_1\,(j\omega)^1 + \alpha_0\,(j\omega)^0} \tag{3}$$

This can be rewritten:

$$\frac{X_p(\omega)}{F_q(\omega)} = \frac{\sum_{k=0}^{n} \beta_k\,(j\omega)^k}{\sum_{k=0}^{m} \alpha_k\,(j\omega)^k} \tag{4}$$

Further rearranging yields the following equation that is linear in the unknown α and β terms:

$$\sum_{k=0}^{m} \alpha_k\,(j\omega)^k X_p(\omega) = \sum_{k=0}^{n} \beta_k\,(j\omega)^k F_q(\omega) \tag{5}$$

The unknowns in the above linear equation are the scalar alpha and beta coeficients.

Modal Parameter Estimation - MDOF

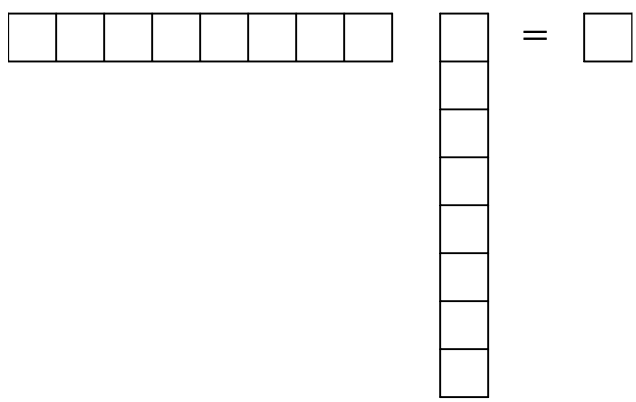
The previous equation can be rearranged into the following matrix form by moving all of the terms to the left:

$$[\alpha_0\ \alpha_1 \cdots \alpha_m\ \beta_0\ \beta_1 \cdots \beta_n\ ] \begin{bmatrix} (j\omega)^0\, X_p(\omega) \\ (j\omega)^1\, X_p(\omega) \\ \cdots \\ (j\omega)^m\, X_p(\omega) \\ -(j\omega)^0\, F_q(\omega) \\ -(j\omega)^1\, F_q(\omega) \\ \cdots \\ -(j\omega)^n\, F_q(\omega) \end{bmatrix} = 0 \tag{6}$$

The above linear equation involves $(m+1)+(n+1)$ ***unknowns. Since any unknown can be assumed to be unity (normally one of the alpha coefficients), the number of unknowns can be reduced by one to*** $m+n+1$***. The choice of which unknown that is set to unity is a numerical issue.***

Modal Parameter Estimation - MDOF

$$[\alpha_0\ \alpha_1 \cdots \alpha_m\ \beta_0\ \beta_1 \cdots \beta_n\,] \begin{bmatrix} (j\omega)^0\, X_p(\omega) \\ (j\omega)^1\, X_p(\omega) \\ \cdots \\ (j\omega)^m\, X_p(\omega) \\ -(j\omega)^0\, F_q(\omega) \\ -(j\omega)^1\, F_q(\omega) \\ \cdots \\ -(j\omega)^n\, F_q(\omega) \end{bmatrix} = 0 \qquad (7)$$

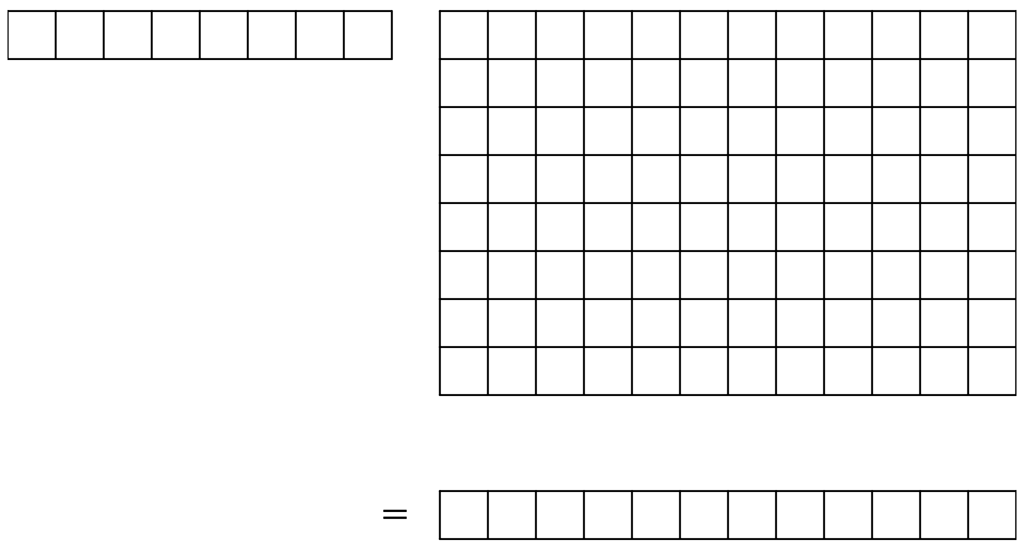


The unknowns in the above equation can be solved by forming at least $m+n+1$ equations by acquiring input and output data at least $m+n+1$ frequencies. Generally, more equations are used by using more frequency information, therefore, requiring a least squares solution for the unknowns.

Modal Parameter Estimation - MDOF

Overdetermined Linear Equations

Additional Frequency Information



Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Single Input - Single Output FRF Model

$$H_{pq}(\omega) = \frac{\beta_n\,(j\omega)^n + \beta_{n-1}\,(j\omega)^{n-1} + \cdots + \beta_1\,(j\omega)^1 + \beta_0\,(j\omega)^0}{\alpha_m\,(j\omega)^m + \alpha_{m-1}\,(j\omega)^{m-1} + \cdots + \alpha_1\,(j\omega)^1 + \alpha_0\,(j\omega)^0} \tag{8}$$

This can be rewritten:

$$H_{pq}(\omega) = \frac{\sum_{k=0}^{n} \beta_k\,(j\omega)^k}{\sum_{k=0}^{m} \alpha_k\,(j\omega)^k} \tag{9}$$

Further rearranging yields the following equation that is linear in the unknown α and β terms:

$$\sum_{k=0}^{m} \alpha_k\,(j\omega)^k H_{pq}(\omega) = \sum_{k=0}^{n} \beta_k\,(j\omega)^k \tag{10}$$

The unknowns in the above linear equation are the scalar alpha and beta coeficients.

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Single Input - Single Output FRF Model

The previous equation can be rearranged into the following matrix form by moving all of the terms to the left:

$$[\alpha_0 \; \alpha_1 \cdots \alpha_m \; \beta_0 \; \beta_1 \cdots \beta_n] \begin{bmatrix} (j\omega)^0 \, H_{pq}(\omega) \\ (j\omega)^1 \, H_{pq}(\omega) \\ \cdots \\ (j\omega)^m \, H_{pq}(\omega) \\ -(j\omega)^0 \\ -(j\omega)^1 \\ \cdots \\ -(j\omega)^n \end{bmatrix} = 0 \tag{11}$$

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Multiple Input - Multiple Output Model

The expansion of the previous model to handle multiple input - multiple output data yields the following:

$$\sum_{k=0}^{m}\left[[\alpha_k]\,(j\omega)^k\right]\{X(\omega)\} = \sum_{k=0}^{n}\left[[\beta_k]\,(j\omega)^k\right]\{F(\omega)\} \qquad (12)$$

The unknowns in the above linear equation are the matrix alpha and beta coeficients. Note that the size of the coefficient matrix $[\alpha_k]$ ***will normally be*** $N_o \times N_o$ ***and the coefficient matrix*** $[\beta_k]$ ***will normally be*** $N_o \times N_i$ ***when the equations are developed from experimental data.***

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Multiple Input - Multiple Output Model

The previous equation can be rearranged into the following matrix form by moving all of the terms to the left:

$$[[\alpha_0][\alpha_1]\cdots[\alpha_m][\beta_0][\beta_1]\cdots[\beta_n]]\begin{bmatrix}(j\omega)^0\,\{X(\omega)\}\\(j\omega)^1\,\{X(\omega)\}\\\cdots\\(j\omega)^m\,\{X(\omega)\}\\-(j\omega)^0\,\{F(\omega)\}\\-(j\omega)^1\,\{F(\omega)\}\\\cdots\\-(j\omega)^n\,\{F(\omega)\}\end{bmatrix}=0 \tag{13}$$

The above linear equation represents N_o linear equations involving matrix unknowns. Since any unknown alpha matrix can be assumed to be the identity matrix $[I]$, the number of unknowns can be reduced by one. The choice of which unknown alpha matrix that is set to identity is a numerical issue.

These unknowns can be solved by forming sufficient equations by acquiring input and output data at a number of frequencies.

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Multiple Input - Multiple Output FRF Model

The previous multiple input - multiple output model can be reformulated to utilize frequency response function (FRF) data as follows. First, post multiply both sides of the equation by $\{F\}^H$***:***

$$\sum_{k=0}^{m} \left[[\alpha_k] \, (j\omega)^k \right] \{X(\omega)\} \; \{F(\omega)\}^H = \sum_{k=0}^{n} \left[[\beta_k] \, (j\omega)^k \right] \{F(\omega)\} \; \{F(\omega)\}^H \tag{14}$$

Now recognize that the product of $\{X(\omega)\} \, \{F(\omega)\}^H$ ***is the output-input cross spectra matrix (***$[G_{xf}(\omega)]$***) for one ensemble and*** $\{F(\omega)\} \, \{F(\omega)\}^H$ ***is the input-input cross spectra matrix (***$[G_{ff}(\omega)]$***) for one ensemble. With a number of ensembles (averages), these matrices are the common matrices used to estimate the FRFs in a MIMO case. This yields the following cross spectra model:***

$$\sum_{k=0}^{m} \left[[\alpha_k] \, (j\omega)^k \right] \left[G_{xf}(\omega) \right] = \sum_{k=0}^{n} \left[[\beta_k] \, (j\omega)^k \right] \left[G_{ff}(\omega) \right] \tag{15}$$

The previous cross spectra model can be reformulated to utilize frequency response function (FRF) data by post multiplying both sides of the equation by $[G_{ff}(\omega)]^{-1}$***:***

$$\sum_{k=0}^{m} \left[[\alpha_k] \, (j\omega)^k \right] \left[G_{xf}(\omega) \right] \left[G_{ff}(\omega) \right]^{-1} = \sum_{k=0}^{n} \left[[\beta_k] \, (j\omega)^k \right] \left[G_{ff}(\omega) \right] \left[G_{ff}(\omega) \right]^{-1} \tag{16}$$

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Multiple Input - Multiple Output FRF Model

Therefore, the multiple input - multiple output FRF model is:

$$\sum_{k=0}^{m} \left[[\alpha_k]\,(j\omega)^k \right] [H(\omega)] = \sum_{k=0}^{n} \left[[\beta_k]\,(j\omega)^k \right] [I] \tag{17}$$

The unknowns in the above linear equation are the matrix alpha and beta coeficients. Note that the size of the coefficient matrix $[\alpha_k]$ will normally be $N_o \times N_o$ and the coefficient matrix $[\beta_k]$ will normally be $N_o \times N_i$ when the equations are developed from experimental data.

Since the FRF matrix is normally considered to be reciprocal ($H_{pq} = H_{qp}$), the previous formulation can be developed from the transposed FRF matrix. This means that the the size of the coefficient matrix $[\alpha_k]$ will be $N_i \times N_i$ and the coefficient matrix $[\beta_k]$ will normally be $N_i \times N_o$ for this case.

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Frequency Domain

Multiple Input - Multiple Output FRF Model

The previous equation can be rearranged into the following matrix form by moving all of the terms to the left:

$$[[\alpha_0][\alpha_1]\cdots[\alpha_m][\beta_0][\beta_1]\cdots[\beta_n]]\begin{bmatrix}(j\omega)^0\,[H(\omega)]\\(j\omega)^1\,[H(\omega)]\\\cdots\\(j\omega)^m\,[H(\omega)]\\-(j\omega)^0\,[I]\\-(j\omega)^1\,[I]\\\cdots\\-(j\omega)^n\,[I]\end{bmatrix}=0 \qquad (18)$$

The above linear equation represents N_o linear equations involving matrix unknowns. Since any unknown alpha matrix can be assumed to be the identity matrix $[I]$, the number of unknowns can be reduced by one. The choice of which unknown alpha matrix that is set to identity is a numerical issue.

These unknowns can be solved by forming sufficient equations by acquiring input and output data at a number of frequencies.

Modal Paramter Estimation - MDOF

Fundamental Identification Models - Time Domain

Single Input - Single Output Model

The time domain model that corresponds to the single input - single output frequency domain model is known as an auto-regressive, moving average (ARMA) model as follows:

$$\sum_{k=0}^{m} \alpha_k \; x(t_{i+k}) = \sum_{k=0}^{n} \beta_k \; f(t_{i+k}) \tag{19}$$

Using the same approach as in the frequency domain yields the following matrix equation:

$$[\alpha_0 \, \alpha_1 \cdots \alpha_m \; \beta_0 \, \beta_1 \cdots \beta_n] \begin{bmatrix} x(t_{i+0}) \\ x(t_{i+1}) \\ \cdots \\ x(t_{i+m}) \\ -f(t_{i+0}) \\ -f(t_{i+1}) \\ \cdots \\ -f(t_{i+n}) \end{bmatrix} = 0 \tag{20}$$

Modal Paramter Estimation - MDOF

Fundamental Identification Models - Time Domain

Single Input - Single Output Model

The same frequency domain discussion concerning the dimension and solution of the above equation applies, noting that the above equation is real valued. The other important consideration is to note that the above linear equation can be solved for the $(m+1)+(n+1)$ ***unknowns by simply taking additional time domain input and output data at discrete time steps.***

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Time Domain

Single Input - Single Output IRF Model

If the discussion is limited to the use of free decay or impulse response function data, the previous time domain equations can be simplified by noting that the forcing function can be assumed to be zero for all time greater than zero. If this is the case, the $[\beta_k]$ coefficients can be eliminated from the equations.

$$\sum_{k=0}^{m} \alpha_k \, h_{pq}(t_{i+k}) = 0 \tag{21}$$

Using the same approach as in the frequency domain yields the following matrix equation:

$$[\alpha_0 \, \alpha_1 \cdots \alpha_m] \begin{bmatrix} h_{pq}(t_{i+0}) \\ h_{pq}(t_{i+1}) \\ \cdots \\ h_{pq}(t_{i+m}) \end{bmatrix} = 0 \tag{22}$$

Note the above equation is real-valued and has considerably fewer unknowns (no beta coefficients when compared to the corresponding frequency domain FRF model). This is a significant numerical difference in the two formulations.

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Time Domain

Multiple Input - Multiple Output Model

The time domain equation that corresponds to the multiple input - multiple output frequency domain model follows:

$$\sum_{k=0}^{m} [\alpha_k] \{ x(t_{i+k}) \} = \sum_{k=0}^{n} [\beta_k] \{ f(t_{i+k}) \} \tag{23}$$

Using the same approach as in the frequency domain yields the following matrix equation:

$$[[\alpha_0]\,[\alpha_1]\cdots[\alpha_m]\,[\beta_0]\,[\beta_1]\cdots[\beta_n]] \begin{bmatrix} \{x(t_{i+0})\} \\ \{x(t_{i+1})\} \\ \cdots \\ \{x(t_{i+m})\} \\ -\{f(t_{i+0})\} \\ -\{f(t_{i+1})\} \\ \cdots \\ -\{f(t_{i+n})\} \end{bmatrix} = 0 \tag{24}$$

The same frquency domain discussion concerning the dimension and solution of the above equation applies, noting that the above equation is real valued.

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Time Domain

Multiple Input - Multiple Output Model

If the discussion is again limited to the use of free decay or impulse response function data, the $[\beta_k]$ coefficients can be eliminated from the equations.

$$\sum_{k=0}^{m} [\alpha_k]\,[h(t_{i+k})] = 0 \tag{25}$$

Using the same approach as in the frequency domain yields the following matrix equation:

$$[[\alpha_0]\,[\alpha_1]\cdots[\alpha_m]]\begin{bmatrix}[h(t_{i+0})]\\ [h(t_{i+1})]\\ \cdots\\ [h(t_{i+m})]\end{bmatrix} = 0 \tag{26}$$

Note the above equation is real-valued and has considerably fewer unknowns (no beta coefficients when compared to the corresponding frequency domain FRF model). This is a significant numerical difference in the two formulations. Note that the size of the alpha coefficient matrices $[\alpha_k]$ will normally be $N_o \times N_o$ when the equations are developed from experimental data (typically, Fourier transform of the FRF data).

Modal Parameter Estimation - MDOF

Fundamental Identification Models - Summary of First Stage

Matrix Coefficient Polynomials

Time Domain: Impulse Response Function

$$\sum_{k=0}^{m} [\alpha_k] \left[h(t_{i+k}) \right] = 0$$

$$|\, [\alpha_m]\, z^m + [\alpha_{m-1}]\, z^{m-1} + [\alpha_{m-2}]\, z^{m-2} + \ldots\ldots\ldots + [\alpha_0] \,| = 0$$

$$z_r = e^{\lambda_r \Delta t} \qquad \lambda_r = \sigma_r + j\ \omega_r$$

$$\sigma_r = \mathrm{Re}\left[\frac{ln\ z_r}{\Delta t} \right] \qquad \omega_r = \mathrm{Im}\left[\frac{ln\ z_r}{\Delta t} \right]$$

Frequency Domain: Frequency Response Function

$$\sum_{k=0}^{m} \left[(j\omega_i)^k\, [\alpha_k] \right] \left[H(\omega_i) \right] = \sum_{k=0}^{n} \left[(j\omega_i)^k\, [\beta_k] \right] [I]$$

$$[\alpha_m]\, s^m + [\alpha_{m-1}]\, s^{m-1} + [\alpha_{m-2}]\, s^{m-2} + \ldots\ldots\ldots + [\alpha_0] = 0$$

$$s_r = \lambda_r = \sigma_r + j\ \omega_r$$

Modal Parameter Estimation - MDOF

Second Stage - Residue Estimation

Once the modal frequencies are known, the solution for the residues comes from another overdetermined set of equations, typically from the partial fraction model.

Single Reference:

$$H_{pq}(\omega) = \sum_{r=1}^{N} \frac{A_{pqr}}{j\omega - \lambda_r} + \frac{A^*_{pqr}}{j\omega - \lambda^*_r}$$

$$\{H(\omega)\}_q = \sum_{r=1}^{N} \frac{\{A\}_{qr}}{j\omega - \lambda_r} + \frac{\{A^*\}_{qr}}{j\omega - \lambda^*_r}$$

Multiple Reference:

$$[H(\omega)\ ] = \sum_{r=1}^{N} \frac{\left[A_r \right]}{j\omega - \lambda_r} + \frac{\left[A^*_r \right]}{j\omega - \lambda^*_r}$$

$$[H(\omega)\ ] = \sum_{r=1}^{2N} \frac{\left[A_r \right]}{j\omega - \lambda_r}$$

$$[H(\omega)]_{N_o \times N_i} = \left[\psi \ \right]_{N_o \times 2N} \left\lceil \frac{1}{j\omega - \lambda_r} \right\rfloor_{2N \times 2N} [L\ ]^T{}_{2N \times N_i}$$

Modal Parameter Estimation - MDOF

Primary Issue - What is Model Order?

The meaning of *model order* ***can be confusing based upon different algorithmic developments since model order can have several different specific meanings. For the purposes of further discussion, model order will refer to the mathematical order of the highest differential or polynomial term in the matrix, or scalar, coefficient model equation. Therefore, the number of modal frequencies (roots) that can be determined will be equal to the model order (***m***) times the size of the square matrix coefficients (for example,*** N_o***) in the matrix (***$N_o \times N_o$***) model.***

For the practical case with measured data, the number of modal frequencies is unknown or known only in some bounded fashion. *Model order determination* ***involves the engineering process used to decide how many modal frequencies are present in the data and, therefore, what model order to use.***

Modal Parameter Estimation - MDOF

Model Order Determination

- ***FRF Waterfall***
- ***Automoment of the FRF Matrix***
- ***Mode Indication Functions.***
    - ***MultiVariate Mode Indicator Function (MvMIF)***
    - ***Complex Mode Indicator Function (CMIF)***
- ***Rank Estimation.***
- ***Consistency (Stability) Diagram.***
- ***Realistic Pole Location***
- ***Pole Density Map***
- ***Measurement Synthesis and Comparison (Curve-Fit).***

Modal Parameter Estimation - MDOF

Algorithmic Differences

Many of the differences in the algorithms that are available involve the mannare in whaich the data is handled. This is very important relative to numerical issues or weighting of the data and does affect the modal parameters that are estimated.

- ***Data Filtering: Data limited within minimum and maximum temporal axis values.***
- ***Data Sieving: Data limited to prescribed input DOFs and/or output DOFs.***
- ***Data Decimation: Data limited by removing selected frequencies or time points.***
- ***Coefficient Condensation: Decomposition transformation from physical coordinates (*** N_o ***or*** N_i***) to the approximate number of effective modal frequencies (*** N_e***). Currently, singular value decompositions (SVD) or eigenvalue decompositions (ED) are used.***
- ***Equation Condensation: Equation condensation methods are used to reduce the number of equations based upon measured data to more closely match the number of unknowns in the modal parameter estimation algorithms.***

Modal Parameter Estimation - MDOF

Conclusion/Summary of MDOF Methods

Most modern modal identification algorithms can be outlined as follows:

- ***Load Measured Data into Linear Equation Form.***
  - ***Limit Temporal Information (Filter)***
  - ***Limit Spatial Information (Sieve)***
  - ***Limit Temporal Information (Decimate)***
  - ***Perform Coefficient Condensation (Low Order Models)***
  - ***Perform Equation Condensation***
- ***Find Scalar or Matrix Autoregressive Coefficients (**$[\alpha_k]$**).***
- ***Solve Matrix Polynomial for Modal Frequencies.***
  - ***Formulate Companion Matrix.***
  - ***Obtain Eigenvalues of Companion Matrix. (**λ_r **or** z_r**).***
  - ***Convert Eigenvalues from** z_r **to** λ_r **(time domain only).***
  - ***Obtain Modal Participation Vectors** L_{qr} **or Modal Vectors** $\{\psi\}_r$ **from Eigenvectors of the Companion Matrix.***
- ***Find Residues and Modal Scaling.***

Modal Parameter Estimation

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